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Spectral statistics across the many-body localization transition

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The many-body localization transition (MBLT) between ergodic and many-body localized phases in disordered interacting systems is a subject of much recent interest. The statistics of eigenenergies is known to be a powerful probe of crossovers between ergodic and integrable systems in simpler examples of quantum chaos. We consider the evolution of the spectral statistics across the MBLT, starting with mapping to a Brownian motion process that analytically relates the spectral properties to the statistics of matrix elements. We demonstrate that the flow from Wigner-Dyson to Poisson statistics is a two-stage process. First, a fractal enhancement of matrix elements upon approaching the MBLT from the delocalized side produces an effective power-law interaction between energy levels, and leads to a plasma model for level statistics. At the second stage, the gas of eigenvalues has local interactions and the level statistics belongs to a semi-Poisson universality class. We verify our findings numerically on the XXZ spin chain. We provide a microscopic understanding of the level statistics across the MBLT and discuss implications for the transition that are strong constraints on possible theories.

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Introduction. Quantum and statistical mechanics represent two seemingly rather different approaches to the description of complex physical systems. Yet these two viewpoints agree for a wide class of isolated quantum systems, which are said to thermalize [1,2]. Determining the circumstances under which an isolated quantum many-body system becomes its own thermal bath and thermalizes itself, just as Baron Munchausen could pull himself out of a mire by his own hair, perhaps using some kind of fluctuation, is an open question.

Phenomena similar to the emergence of thermalization also occur in few-body quantum systems, which frequently show the emergence of so-called quantum chaos [3]. There, upon changing parameters/number of degrees of freedom, the classical system can go from regular to chaotic behavior. On a quantum level this results in changes of the level statistics, which has proven to be a powerful probe of the system properties in the context of quantum chaos. In particular, there exist two standard universal limits: the Poisson statistics (PS) and the Wigner-Dyson level statistics (WDS) [4]. For few-body systems, PS applies to systems that are classically integrable and do not have any level repulsion. WDS stems from random-matrix theory and holds for generic chaotic systems, where energy levels repel each other (i.e., the energy difference between neighboring levels is statistically unlikely to be small compared to the mean level spacing).

Integrable (nonchaotic) behavior is abundant in the context of few-body physics. On the other hand, in the many-body world, the only nonthermalizing *phase* (in the sense of stability to small perturbations) is represented by many-body localized (MBL) systems [5,6]. Recent progress established that thermalization fails in the MBL phase due to the existence of extensively many conserved quantities [7–10]. On the other hand, it is known that one can tune the system through a phase transition into a thermalizing ergodic phase [11–20]. Below, we aim to understand the evolution of the level statistics across the MBL-to-ergodic transition, gaining insights into the breakdown of thermalization.

The crossover between PS and WD statistics has been studied extensively in a single-particle physics context: for a quantum kicked rotor [21], integrability breaking perturbations [22,23], and single-particle Anderson localization transitions (ALT) [24–26]. In the many-body problems, a PS to WD crossover is also known to occur upon breaking of (quantum) integrability [27]. In most of the examples, the PS and WDS are the only two stable points. The only known exception is the ALT, where a *universal statistics* different from PS and WDS emerges at the mobility edge [24].

The spectral statistics in the case of MBL transition was demonstrated to evolve from WDS to PS as one localizes the system [11,28–30], however, not much is known about the intermediate statistics. The common probe used to characterize the level statistics across MBLT is an average ratio of the consecutive energy spacings [11–13,18]. However, this is a single parameter and it does not provide much insight into the intermediate form of the level statistic, nor into the physical details of its crossover.

In this paper, we study how the spectral statistics changes across the MBL-delocalization transition. In order to build a microscopic understanding of the level statistics, we generalize Dyson's Brownian motion model [31], previously applied to the ALT [32], to the many-body case. From the mapping to Brownian motion, we obtain nontrivial relations between the fractality [17–20], the spectral statistics, and the properties of matrix elements across the MBLT [20,33]. While many features can be simultaneously explained in this analysis, one surprise is that there appear to be two different regimes of intermediate spectral statistics: in one, the effective interaction between energy levels in the plasma model has a variable power law, while in the other, the effective interaction is short-ranged but over a variable number of levels.

Within the picture of Brownian motion [31,32], the level statistics is controlled by the effective interaction between energy levels, see Fig. 1. In particular, deep in the ergodic phase, the WD statistics emerges from the partition function of a one-dimensional Coulomb gas, where particles interact

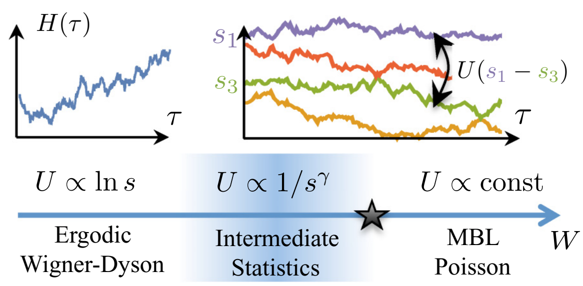

FIG. 1. (Top) Random walk in a space of Hamiltonians induces a stochastic process on the eigenenergies. The interaction between eigenlevels is set by a potential energy $U(s_i - s_j)$. (Bottom) The evolution of the interaction between levels $U(s)$ across the MBL transition determines the level statistics.

with a logarithmic potential $U(s) = -\ln|s|$. At a first stage, upon approaching the MBL transition, the effective interaction starts to decay as a power law: $U(s_i - s_j) = |s_i - s_j|^{-\gamma}$ when $|s_1 - s_2| \geqslant N_{\text{erg}}$. The power-law interaction changes the tails of the level statistics, so it can be approximately described by the plasma model, and is intermediate between the PS and the WDS case. At the second stage, when the exponent γ becomes bigger than one, the interaction becomes effectively short-ranged, and the level spacing distribution tends to the semi-Poisson distribution [34]. In this regime, it is the range of the interaction which changes with the disorder/system size. As soon as the range of interactions reaches zero, we arrive at Poisson statistics.

Before discussing the implications of the above picture of the level statistics, we justify the proposed cartoon using both analytic and numeric arguments. In particular, we argue that the parameter γ introduced above can be extracted from the properties of the many-body matrix elements, which decay as a power-law with energy separation between eigenstates, where $\gamma \leqslant 1$ is the same power that controls the level statistics. The power-law behavior of matrix elements can be viewed as a generalization of the Chalker-Daniell scaling of wave function overlap [35] to the many-body case, and it is consistent with the fractality of the wave functions near MBLT [17–20].

Plasma model for level correlations. In the random matrix theory, the joint probability density for the random matrix ensembles reads

$$P(\{s_i\}) = \frac{e^{-\beta H}}{Z},\ H = \sum_i W(s_i) + \sum_{i<j} U(s_i - s_j), \quad (1)$$

where $\beta = 1$ for the orthogonal matrix ensemble, which will be of primary interest. The confining potential $W(s) = s^2/2$ is parabolic, and the interaction is $U(s_i - s_j) = -\ln|s_i - s_j|$. As Dyson demonstrated in his pioneering work [31], this distribution function may be viewed as a stationary distribution of the stochastic random walk in a space of matrices (Hamiltonians).

To derive the joint distribution of eigenenergies from a random walk, one can start from the eigenbasis and perform a stochastic step in the space of Hamiltonians, induced by ΔH. Then, we get the energy correction in a form

$$\Delta s_n = V_{nn} + \sum_{m\neq n} \frac{V_{mn}V_{nm}}{s_n - s_m}, \quad V_{mn} = \langle m|\Delta H|n\rangle, \quad (2)$$

which is the shift of eigenenergies induced by the perturbation ΔH up to second order. For Gaussian ensembles of random matrices, using $\langle V_{nm}V_{mn}\rangle = \frac{2}{\beta}\Delta\tau$ and $\langle V_{nn}V_{mm}\rangle = \delta_{mn}\Delta\tau$, one can derive the Fokker-Planck equation (see Ref. [36] for more details). Its stationary (equilibrium) solution is given by Eq. (1) with a logarithmic interaction. Note that in what follows we omit the damping term, which keeps the bandwidth fixed [36].

Dyson's mapping was generalized to the case of disordered problems [32]. For such problems, it is natural to perform a random walk (RW) in a space of Hamiltonians by changing realizations of disorder. As we are going to concentrate on the properties of a spin chain of L spins in a random magnetic field, which is coupled to the z component of a spin S_i^z, we take $\Delta H = \sum_{i=1}^{L} h_i(\tau)S_i^z$, with $\langle h_i(\tau)h_j(\tau')\rangle = v^2\delta(\tau - \tau')\delta_{ij}$. Similar to the case of random matrices [3,31,36], the two correlators, which determine the level dynamics, are

$$\langle V_{nn}V_{mm}\rangle = \delta d_{nm} = \langle n|S_i^z|n\rangle\langle m|S_i^z|m\rangle, \quad (3)$$

$$\langle V_{nm}V_{mn}\rangle = \delta c_{nm} = \left|\langle m|S_i^z|n\rangle\right|^2, \quad (4)$$

where we assumed that $v^2 = \delta/L$, where δ is the many-body level spacing, so that s_n represent the unfolded energy spectrum. The correlator (3) sets the spectrum of a random noise, while the spectral function c_{nm} determines the interaction between levels in the ensemble.

Effective interaction between levels. The RW process depends crucially on two correlators (3) and (4). To make analytic progress, we use a mean-field-like approximation [32], assuming that d_{nm} and c_{nm} can be replaced by their ensemble averages,

$$c(\omega) = \langle c_{nm}\delta(s_n - s_m - \omega)\rangle, \quad (5)$$

(and similar expression for d_{nm}) which now depend only on the energy difference between eigenstates. For the single-particle Anderson localization, the c_{nm} and d_{nm} necessarily coincide with the wave function overlaps [32], $c_{nm} = d_{nm} \propto \int dx|\psi_n(\tau,x)|^2|\psi_m(\tau,x)|^2$. The fractality of the wave function near the mobility edge results in a power-law enhancement of $c(\omega) \propto A/\omega^\gamma$ [35,37]. In the case of ALT, this enhancement arises because the envelope of wave functions nearby in energy lives on the same multifractal domain [37]. In the many-body case, similar enhancement can arise from the fractal structure of the wave function in the Hilbert space in a vicinity of MBLT [17–20].

We view the matrix elements $V_{nm} = \langle n|S_i^z|m\rangle$ as coefficients of the wave function of excitation created by a local operator S_i^z from an eigenstate $|m\rangle$ [20]. We assume that the inverse participation ratio (IPR) $I_2 = \mathcal{V}\sum_m |V_{nm}|^4 \propto \mathcal{V}^{-d_2}$, where d_2 is a generalized fractal dimension, and $\mathcal{V} = \exp(sL)$ is the number of states in the Hilbert space. We translate the

IPR into scaling with an energy separation as $\mathcal{V}^2\langle V_{nn}^2 V_{nk}^2\rangle \propto (\mathcal{V}/\mathcal{R})^{1-d_2}$, where $\mathcal{R} = |n-k| \approx (E_k - E_n)/\delta$. From here, omitting the diagonal matrix element V_{nn} given by the spin expectation value, we arrive to the scaling

$$c(\omega) \propto \left(\frac{J}{\omega}\right)^{\gamma}, \qquad \gamma = 1 - d_2. \tag{6}$$

The above argument should be viewed as phenomenological; at present, the microscopic nature of a fractal behavior is not clear, although Griffiths (rare-region) effects [18] in the vicinity of an MBL transition provide one possible microscopic scenario. Also, relating d_2 to the properties of matrix elements, i.e., exponent κ in the scaling [20,33], $|V_{nm}| \propto \exp(-(s+\kappa)L)$ is an interesting question.

The correlation between the diagonal matrix elements, the function d_{nm}, also shows a power-law dependence. However, there is an enhancement of d_{nm} for $n = m$, allowing to approximate $d(\omega)$ as a delta-function (see Supplemental Material for additional discussion [36]).

Implications for spectral statistics. Using the power-law form of $c(\omega)$, Eq. (6), and the delta-function form of $d(\omega)$, we can map our model onto the plasma model for the level statistics [38], provided $\gamma < 1$. The plasma model assumes a power-law interaction potential $U(s) = A/|s|^{\gamma}$ in the joint distribution function (1). It predicts the tails of the level statistics $P(s) \propto s^{\beta}\exp(-h_{\gamma}s^{2-\gamma})$ for $s \gg 1$, and the variance of the number of levels in a box of size N becomes $\operatorname{var} N \propto N^{\gamma}$, which is intermediate between a WD-like rigidity $\operatorname{var} N \propto \ln N$ and the Poisson case [3,4].

For larger values of $\gamma \geqslant 1$, the effective interaction in the gas of eigenvalues becomes short ranged, and mapping to the plasma model no longer works. Instead, spectral properties now are expected to be well-described by a family of semi-Poisson distributions [34], which arise from a gas of eigenvalues with a finite-range interaction. They predict a Poisson-like behavior of the tails of $P(s)$ and level compressibility $P(s) \propto s^{\beta}e^{-(\beta h+1)s}$, and $\operatorname{var} N \propto \chi N$ with $\chi \leqslant 1$, where h is the range of interactions. Such level statistics has been dubbed "critical" in the literature [39–41] and is believed to describe the level statistics at the ALT [25,26].

Using the above intuition, we propose the following form of the level spacing distribution and spectral rigidity to interpolate between WDS and PS,

$$P(s;\beta,\gamma_P) = C_1 s^{\beta}\exp\left(-C_2 s^{2-\gamma_P}\right), \quad \operatorname{var} N = \chi N^{\gamma_{\mathrm{var}}}, \tag{7}$$

where the parameter $1 \geqslant \gamma_P,\gamma_{\mathrm{var}} \geqslant 0$ controls the tails of the statistics and level rigidity, and $1 \geqslant \beta \geqslant 0$ determines the level repulsion. The constants $C_{1,2}$ can be fixed by requiring that $\langle 1\rangle = \langle s\rangle = 1$. When $\gamma_P = 0$, this distribution becomes WD. In the opposite limit, $\gamma_P \to 1$, distribution (7) becomes a semi-Poission with generic β. For the spectral rigidity, our interpolating function also can describe the (semi-)Poisson limit, however, failing to capture the logarithmic growth of $\operatorname{var} N$ in the WD case.

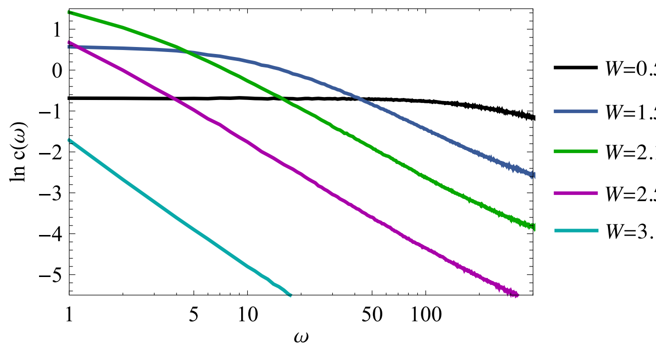

FIG. 2. Averaged function $c(E)$ evolves from being almost flat at low disorder ($W = 0.5$) to a power-law decay. Note that for the intermediate values of disorder, the matrix element is enhanced at small energy differences compared to the limit of weak disorder.

Numerical results. We use the XXZ spin chain in a random field as a specific model with a previously located MBL transition [11] to test our picture of level statistics. The Hamiltonian is

$$\hat{H}_{\mathrm{XXZ}} = \sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j + \sum_i w_i S_i^z, \quad S^{x,y,z} = \frac{1}{2}\sigma^{x,y,z}, \tag{8}$$

where disorder enters via random fields w_i uniformly distributed in the interval $[-W;W]$. We perform exact diagonalization for chains of size $L = 12,\ldots,16(18)$ with periodic boundary conditions to extract the properties of the matrix elements (spectral statistics). We use the central part of the many-body spectrum, which corresponds to energy density $\varepsilon = (E - E_{\min})/(E_{\max} - E_{\min}) = 0.45 \pm 0.1$ and contains 246, 969, 3794, 14316 levels on average for $L = 12,\ldots,18$. The MBL transition at this energy density is believed to occur near $W_c \approx 3.6$ [13]. To unfold levels, we fit the staircase function with a third-order polynomial. We use both local and global level unfolding schemes [42].

We start by discussing the numerical results for averaged $c(\omega)$, presented in Fig. 2(a). Upon increasing disorder, we see the crossover of $c(\omega)$ from a constant to a power-law decay. As one may expect, this crossover happens at some scale, N_{erg}, so that $c(\omega < N_{\mathrm{erg}}) \propto$ const, and decays as a power-law beyond $\omega > N_{\mathrm{erg}}$. The additional scale N_{erg} has a meaning similar to the correlation length, over which ergodicity holds. As $N_{\mathrm{erg}} \to 0$, the interaction between levels becomes critical even for the smallest separations.

From the power-law form of $c(\omega)$, we expect that level spacing distribution for the XXZ spin chain to be well described by Eq. (7). Figure 3(a) illustrates that the flow of the level statistics is indeed well captured by Eq. (7). We also considered a number of single-parameter ansatz (in particular, Brody and semi-Poisson distribution); none of those could capture the changes of $P(s)$ across MBLT. The $P(s)$ for disorder $W < 2$ is not shown, as it looks very similar to the WD distribution: since $P(s)$ is influenced the most by the interaction between close levels, N_{erg} must become close to zero before we see the flow in the level statistics. In contrast to the level statistics, which is influenced by a noncritical part of $c(\omega)$, the spectral rigidity is expected to be less sensitive

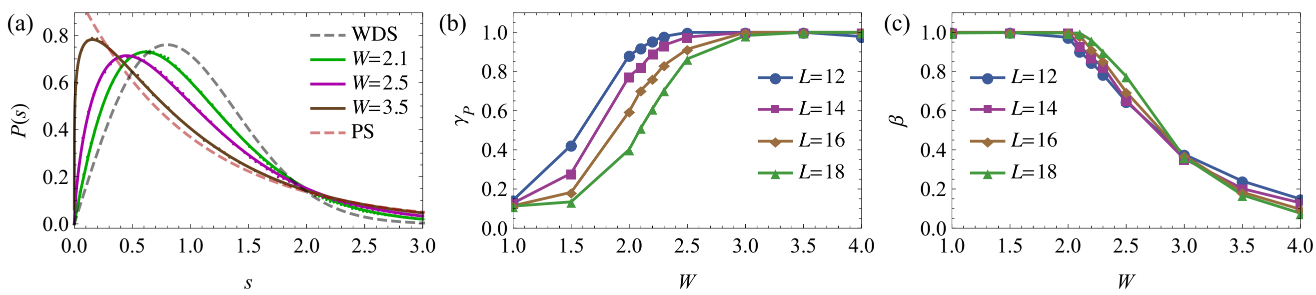

FIG. 3. (a) Evolution of level spacing distributions as the system is tuned towards the MBL phase. Points represent data for $L = 16$, while solid lines are best fits with a two-parameter distribution (7). Red and black dashed lines correspond to Poisson and Wigner-Dyson distributions. (b) The exponent γ_P, controlling the tails of the level statistics, flows with L for $W \lesssim 2.5$, but is constant in the vicinity of MBLT $W_c \approx 3.6$. (c) In contrast, β, controlling the level repulsion, remains constant for $W \lesssim 2$, and starts to flow closer to the MBLT.

to the behavior of $c(\omega)$ at small ω. In SM [36] we show that $\operatorname{var} N$ behaves as a power-law (7), and becomes linear for $W \gtrsim 2$. Also, we test that different estimates for the exponent γ show a reasonable agreement as follows from the plasma model.

Finally, we consider the flow of parameters γ_P and β with increasing system size, presented in Figs. 3(b) and 3(c). While γ_P controlling the tails of the level statistics has a strong flow at disorder $W \leqslant 2.5$, at larger disorders, γ_P is very close to one and changes little with L. This further supports the conclusion that for $W \geqslant 2.5$ the effective interaction between energy levels becomes short-ranged for the largest accessible system sizes. Consistent with our expectation, β shown in Fig. 3(c) changes weakly when the statistics is described by the plasma model ($W \leqslant 2$), and begins to flow once level interactions are local.

Discussion and open questions. Using analytical and numerical arguments, we described the spectral properties across the MBL transition using a two-stage flow picture. Note that we need at least two parameters, γ and N_{erg}, to describe the level statistics. This is not surprising if we recall that even in the case of ALT, the existence of multifractality means that to describe the universal properties one requires more information beyond the small number of critical indices needed for a simple thermodynamic phase transition [25,26]. Below, we discuss the implications of the proposed picture of the spectral statistics flow.

At the first stage, the “correlation length” N_{erg} shrinks to zero, but the exponent responsible for level interactions γ is smaller than one. Intuitively, the levels beyond the correlation length become more and more different, corresponding to a gradual breakdown of the ETH. Here, the level statistics can be described by the effective plasma model. Although this model was proposed some time ago [38], it does not apply in the case of ALT, despite the presence of multifractality near the single-particle mobility edge. Hence, to the best of our knowledge, the present study is the first physical realization of the plasma model.

The second stage begins at $W \geqslant 2.5$, when $\gamma \geqslant 1$ so that interactions between levels are local. Although we cannot exclude the finite size effects, the numerical estimates for the MBL transition at $W_c \approx 3.6$ suggest that *at the MBL transition* interactions between levels are local. Thus we conjecture that the level statistics near and at the MBLT belongs to the same or similar “critical” family as the universal statistics at the ALT [39–41]. This also naturally explains why the average ratio of the level spacing $r = \min(\delta_n,\delta_{n+1})/\max(\delta_n,\delta_{n+1})$ at the MBLT, widely used in the literature [11–13,18], is very close to the value expected from PS.

The semi-Poisson level statistics emerges at the same value of disorder where the boundary of the Griffiths phase was previously identified in the literature [18], $W \approx 2.5$ (Refs. [16,17] report the onset of ergodicity breaking at the same location). The existing theories of the MBLT [14,15] predict extensive entanglement and subdiffusive transport in the ergodic phase. The wide region of critical statistics near transition may be a manifestation of finite size effects (system sizes studied are smaller that diverging correlation length). Indeed, the strong overlaps only between adjacent energy levels imply logarithmic transport [20], predicted at the MBLT [14,15]. On the other hand, the existence of a thermodynamically stable Griffiths phase is another intriguing possibility.

In closing, we have found that Dyson’s mapping of level statistics to Brownian motion allows one to understand the spectral statistics in the MBL transition at least as well as in the ALT for which it was introduced. There are basic differences between the two transitions, e.g., several quantities which are uniquely defined at the ALT allow inequivalent generalizations to the MBLT. There are two steps of the spectral statistics flow, one with long-range interactions (the plasma model) and one with local interactions, and the boundary between the two is found numerically to coincide with the onset of a Griffiths phase and subdiffusive transport. Since the level statistics is known to be the simplest universal probe of the transition to quantum chaos in simpler problems, understanding the origin and universality of the two-step plasma model of level statistics is an important challenge for theories of the MBLT.

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