Theoretical and Mathematical Physics, **154**(2): 270–293 (2008)

NONLINEAR ALGEBRA AND BOGOLIUBOV'S RECURSION

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We give many examples of applying Bogoliubov's forest formula to iterative solutions of various nonlinear equations. The same formula describes an extremely wide class of objects, from an ordinary quadratic equation to renormalization in quantum field theory.

Keywords: quantum field theory, renormalization, nonlinear algebra

1. Introduction

Bogoliubov's forest formula [1]–[5] is a cornerstone of quantum field theory (QFT) used for renormalizations. It was shown in [6] and especially in [7] that the forest formula is in fact a purely algebraic construction devised for solving a broad set of nonlinear equations iteratively; remarkably, the same formula describes an unusually wide class of problems beginning with the solution of an ordinary quadratic equation and ending with the Lamb shift and renormalization of the standard model. The general presentation is [7] is too abstract and does not contain examples. Some examples can be found in [8]–[14], but they all relate to quantum field theory and therefore obscure the simple algebraic and tensor structure of the forest formula. A pedagogical review [15] deals with adequately simple examples, but it does not pay enough attention to algebra and the forest formula. Our purpose here is to provide the needed illustrations of the formalism in [7], and this is a natural addendum to the general presentation of nonlinear algebra in [16].

1.1. The problem. The forest formula provides an iterative solution of the following problem.

• Let $F(T)$ be a *given* function of some variables T, called times or coupling constants. In QFT applications, $F(T)$ is usually an effective action (logarithm of the partition function) of a physical theory regarded as a function of *bare* coupling constants, but this interpretation is inessential for the forest formula per se.

• Let $\mathcal{P}_-$ be a linear projection operator to the linear subspace of *unwanted* functions. In the renormalization theory context, the function $F(T)$ also depends on a cutoff Λ, and functions that are singular in the limit $\Lambda \to \infty$ are *unwanted*. But the concrete interpretation of the projection operator $\mathcal{P}_-$ is again inessential for the forest formula.

• Given $F(T)$ and $\mathcal{P}_-$, we are ready to formulate the *problem*. Can a change of the *arguments* T in $F(T)$ without changing the *shape* of this function put it in the subspace of *wanted* (nonsingular) functions? In other words, the problem is to find a change of variables $T \to \widetilde{T} = T + Q(T)$ such that

$$\mathcal{P}_-\big\{F\big(T + Q(T)\big)\big\} = 0. \tag{1}$$

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 154, No. 2, pp. 316–343, February, 2008. Original article submitted April 2, 2007.

To make the solution of this problem unambiguous, an additional constraint should be imposed that the *counterterm* $Q(T)$ is purely singular, i.e., lies entirely in the space of *singular* functions:

$$\mathcal{P}_+\{Q(T)\} = 0, \tag{2}$$

where $\mathcal{P}_+ = I - \mathcal{P}_-$ is the projection operator to the space of *unwanted* functions.[1]

1.2. Forest formula: Solution of the problem in terms of Feynman diagrams. The forest formula provides an explicit solution of problem (1), (2) in the language of graph theory. The crucial observation made in [7] is that any series in nonnegative powers of the coupling constants T can be represented as a sum over Feynman diagrams with T at the vertices, and this is the most adequate language for tensor-algebra studies [16] (this is clear from the examples below). This means that both $F(T)$ and the counterterm $Q(T)$ can be expanded into sums over graphs:

$$F(T) = \sum_{\Gamma} \widehat{F}(\Gamma) Z(\Gamma \,|\, T), \qquad Q(T) = \sum_{\Gamma} \widehat{Q}(\Gamma) Z(\Gamma \,|\, T), \tag{3}$$

where the basic functions $Z(\Gamma \,|\, T)$ describe the topology of the diagram Γ: $Z(\Gamma \,|\, T)$ is a product of the coupling constants T at the vertices with indices (if any) contracted along the edges. We assume that the projection operators $\mathcal{P}_\pm$ act on the coefficients $\widehat{F}(\Gamma)$ and $\widehat{Q}(\Gamma)$, not on $Z(\Gamma \,|\, T)$; this is the case in most applications.

The iterative solution of problem (1), (2) is now given by the *forest formula*:

$$\widehat{F}(\Gamma/\Gamma)\widehat{Q}(\Gamma) = -\mathcal{P}_-\biggl\{\widehat{F}(\Gamma) + \sum_{\{\gamma_1 \cup \cdots \cup \gamma_k\}} \widehat{F}(\Gamma/(\gamma_1 \cdots \gamma_k))\widehat{Q}(\gamma_1) \cdots \widehat{Q}(\gamma_k)\biggr\}, \tag{4}$$

where $\{\gamma_1 \cup \cdots \cup \gamma_k\}$ are all possible box subgraphs in the graph Γ, $\gamma_i \in \mathcal{B}\Gamma$, i.e., parts of Γ in a collection of nonintersecting "boxes," and $\Gamma/(\gamma_1 \cdots \gamma_k)$ is obtained by contracting all boxes to points. The factor Γ/Γ is just a single vertex with the same external lines as the original Γ; $\widehat{Q}(\Gamma) \neq 0$ and is given by (4) only if $\widehat{F}(\Gamma/\Gamma) \neq 0$. If $\widehat{Q}(\Gamma) \neq 0$ for some graph Γ, then in $F(T)$, there must exist a corresponding coupling constant $\widehat{F}(\Gamma/\Gamma) \neq 0$. We call this rule the "vertex criterion." It plays an important role in further considerations.

We refer to [6], [7] for a discussion of the algebraic theory behind Eq. (4) and also its relation to the group of diffeomorphisms of the moduli space of theories and to its dual Connes–Kreimer Hopf algebra on graphs. We consider many examples and applications of Eq. (4) in what follows.

1.3. The second-level forest formula. In addition to the first-level forest formula (4), there is a next-level Bogoliubov formula [1]–[3], involving sums over embedded box subgraphs (see (9.7) in [7]). The second-level forest formula directly gives the answer for the renormalized effective action $F(\widetilde{T}) = F\bigl(T + Q(T)\bigr) = \sum_{\Gamma} \widehat{F}_{\mathrm{R}}(\Gamma) Z(\Gamma \,|\, T)$:

$$\widehat{F}_{\mathrm{R}}(\Gamma) = \sum_{\mathcal{F}_\Gamma} \prod_{\mathcal{T} \in \mathcal{F}_\Gamma} \Biggl(\overrightarrow{\prod_{\substack{\text{vertices}\\ \text{of } \mathcal{T}}}} \frac{1}{\widehat{F}(\hat{\gamma}_k/\hat{\gamma}_k)} (-\mathcal{P}_-) \widehat{F}\bigl(\hat{\gamma}_k/\bigl(\hat{\gamma}^1_{k+1} \cdots \hat{\gamma}^{s(k)}_{k+1}\bigr)\bigr) \Biggr). \tag{5}$$

The sum is taken over all possible forests $\mathcal{F}_\Gamma$ (the union of trees for all connected components of Γ is called a forest). For some Γ_1 (a connected component of Γ), the tree $\mathcal{T}$ is built as follows:

[1] In fact, this condition is somewhat more complicated: $\mathcal{P}_+\{\widehat{F}(\Gamma/\Gamma)\widehat{Q}(\Gamma)\} = 0$ (see Secs. I.1.4.1 and I.1.4.2 below).

1. We choose a specific sequence of embedded box subgraphs $\{\gamma_n\}$:

$$\gamma_0 = \Gamma_1, \quad \gamma_1 \in \mathcal{B}\Gamma, \quad \gamma_2 \in \mathcal{B}\gamma_1 \subset \mathcal{B}\Gamma, \quad \dots, \quad \gamma_n \in \mathcal{B}\gamma_{n-1} \subset \dots \subset \mathcal{B}\gamma_1 \subset \mathcal{B}\Gamma_1, \tag{6}$$

where the graphs γ_k are not necessarily connected and can contain $s(k)$ connected components $\gamma_k^1, \dots, \gamma_k^{s(k)}$. But for each γ_k^i, there should exist a $\gamma_{k-1}^{j(i)}$ such that $\gamma_k^i \in \mathcal{B}\gamma_{k-1}^{j(i)}$ (condition (6) is understood exactly thus).

2. Using this sequence of embedded box subgraphs or, equivalently, the collection of nonintersecting boxes, we build a decorated rooted tree $\mathcal{T}$. We associate a vertex of the tree with the lower site of each box. Two vertices are connected by a edge if one of the corresponding boxes lies immediately inside the other (i.e., there are no boxes between the two). The root edge ends at the vertex associated with $\gamma_0 = \Gamma$. In terms of graphs, a vertex of the tree is now associated with each γ_k^i, and there is exactly one edge directed down (toward the root, i.e., associated with the neighboring bigger box) connecting γ_k^i to $\gamma_{k-1}^{j(i)}$, and an unrestricted number of edges directed up connecting γ_k to some collection $\gamma_{k+1}^1, \dots, \gamma_{k+1}^{s(k,i)} \subset \gamma_{k+1}$.

The tree product

$$\overrightarrow{\prod_{\substack{\text{vertices}\\ \text{of } \mathcal{T}}}}$$

associates an operator $\big(\widehat{F}\big([\gamma_k^i/\gamma_k^i]\big)\big)^{-1}(-\mathcal{P}_-) \times \widehat{F}\big(\gamma_k^i/\big(\gamma_{k+1}^1 \cdots \gamma_{k+1}^{j(i)}\big)\big)$ with each vertex γ_k^i, while the projection operator $\mathcal{P}_-$ acts upward along the branches of the tree. The root vertex γ_0 (i.e., a connected component of Γ) contributes just $\widehat{F}\big(\gamma_0/\big(\gamma_1^1 \cdots \gamma_1^{s(0)}\big)\big)$. The arrow over the product sign means that the product is ordered along the branches.

Problem (1), (2) implies that the function $F(T)$ and the projection operator $\mathcal{P}_-$ are already given. But the diagram technique must still be introduced. Moreover, this can be done differently for a given $F(T)$ and $\mathcal{P}_-$. We also mention that the iterative nature of the forest formula does not mean that there is a *small* expansion parameter; moreover, in the renormalization theory context, the relevant expansion parameter is large, not small.

I. Algebraic examples

I.1. Quadratic equation: A bivalent vertex and tree diagrams

Our starting example is

$$\begin{aligned} &F(T) = T + \epsilon T^2, \\ &\mathcal{P}_-\big\{f(\epsilon)\big\} = f(\epsilon) - f(0), \end{aligned} \tag{7}$$

i.e., the functions that depend on ϵ are unwanted. Our goal is to eliminate the ϵ dependence by an ϵ-dependent shift of variables $T \to \widetilde{T} + Q_\epsilon(T)$ such that $F(\widetilde{T}) = \widetilde{T} + \epsilon\widetilde{T}^2$ no longer depends on ϵ. Imposing additional constraint (2), we reduce the problem to

$$\widetilde{T} + \epsilon\widetilde{T}^2 = T, \tag{8}$$

and therefore

$$\begin{aligned} \widetilde{T} = \frac{\sqrt{1+4\epsilon T} - 1}{2\epsilon} &= T + \sum_{k=1} \frac{(-1)^k 2^k (2k-1)!!}{(k+1)!} \epsilon^k T^{k+1} = \\ &= T - \epsilon T^2 + 2\epsilon^2 T^3 - 5\epsilon^3 T^4 + 14\epsilon^4 T^5 - \dots . \end{aligned} \tag{9}$$

We now reproduce the same series from forest formula (4). For this, we must first choose a particular diagram technique. Our first choice is a single bivalent vertex $T = T_{ij}$, where the indices i and j take the single value $i = j = 1$ and are initially omitted. The only possible connected Feynman diagrams are chains and circles, both equal to T^n, where n is the number of vertices. Only two diagrams, with $n = 1$ and $n = 2$, contribute to $F(T)$.

I.1.1. Chain diagrams. We first consider chain diagrams:[2]

$$\widehat{F}(-\!\bullet\!-) = 1, \qquad \widehat{F}(-\!\bullet\!-\!\bullet\!-) = \epsilon. \tag{10}$$

Accordingly $F(\Gamma/\Gamma) \neq 0$ only for chain diagrams Γ. By the vertex criterion, this implies that the only nonzero contributions to Q are from chain diagrams. For them, forest formula (4) gives

$$\begin{aligned}
&Q[1] \equiv Q(-\!\bullet\!-) = -\mathcal{P}_-\big\{F(-\!\bullet\!-)\big\} = 0,\\
&Q[2] \equiv Q(-\!\bullet\!-\!\bullet\!-) = -\mathcal{P}_-\big\{F(-\!\bullet\!-\!\bullet\!-)\big\} = -\epsilon,\\
&Q[3] \equiv Q(-\!\bullet\!-\!\bullet\!-\!\bullet\!-) = -\mathcal{P}_-\big\{F(-\!\bullet\!-\!\bullet\!-\!\bullet\!-) + 2F(-\!\bullet\!-\!\bullet\!-)Q(-\!\bullet\!-\!\bullet\!-)\big\} =\\
&\qquad = -2\mathcal{P}_-\big\{F(-\!\bullet\!-\!\bullet\!-)Q(-\!\bullet\!-\!\bullet\!-)\big\} = +2\epsilon^2.
\end{aligned}$$

In the last formula $F(-\!\bullet\!-\!\bullet\!-\!\bullet\!-) = 0$ because of our choice of $F(T)$, and the coefficient 2 in the next term appears because there are two different possibilities for choosing a box subgraph $-\!\bullet\!-\!\bullet\!-$ inside $-\!\bullet\!-\!\bullet\!-\!\bullet\!-$: $\boxed{\bullet\!-\!\bullet}\!-\!\bullet\!-$ and $-\!\bullet\!-\!\boxed{\bullet\!-\!\bullet}$.

Similarly, three box subgraphs $\boxed{\bullet\!-\!\bullet\!-\!\bullet}\!-\!\bullet\!-$, $\boxed{\bullet\!-\!\bullet}\boxed{\bullet\!-\!\bullet}$ and $-\!\bullet\!-\!\boxed{\bullet\!-\!\bullet\!-\!\bullet}$ contribute to the next Q:

$$Q[4] \equiv Q(-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-) = -\mathcal{P}_-\Big\{F(-\!\bullet\!-\!\bullet\!-)\big(2Q(-\!\bullet\!-\!\bullet\!-\!\bullet\!-) + \big[Q(-\!\bullet\!-\!\bullet\!-)\big]^2\big)\Big\} = -5\epsilon^3.$$

We note that the box subgraphs $-\!\bullet\!-\!\boxed{\bullet\!-\!\bullet}\!-\!\bullet\!-$ and $\boxed{\bullet\!-\!\bullet}\!-\!\bullet\!-\!\bullet\!-$ do not contribute, because contraction of boxes produces *three*-vertex diagrams and such contributions would be multiplied by $F(-\!\bullet\!-\!\bullet\!-\!\bullet\!-)$, which vanishes for our choice of $F(T)$. We let $[N]$ denote the N-vertex chain diagram. Then

$$Q[5] = -\mathcal{P}_-\big\{F[2]\big(2Q[4] + 2Q[2]Q[3]\big)\big\} = 14\epsilon^4,$$

and in the general case, (4) implies

$$Q[N] = -\mathcal{P}_-\left\{F[2]\left(\delta_{N,2} + 2Q[N-1] + \sum_{k=2}^{N-2} Q[k]Q[N-k]\right)\right\}, \quad N \geq 3.$$

For the generating function $Q(t) = \sum_{N=2} t^N Q[N]$, we obtain

$$Q(t) = -F[2]\big(t^2 + 2tQ(t) + Q^2(t)\big), \tag{11}$$

where $F[2] = \epsilon$, i.e., we have our familiar formula (9):

$$Q(t) + t + \epsilon(Q(t) + t)^2 = t, \qquad Q(t) + t = \frac{\sqrt{1 + 4\epsilon t} - 1}{2\epsilon}. \tag{12}$$

[2]We herafter omit the hats on $\widehat{Q}(\Gamma)$ and $\widehat{F}(\Gamma)$ because that they are easily distinguished from the functions $Q(T)$ and $F(T)$ of the T variables in the corresponding context.

I.1.2. Introducing coefficients. In the preceding example, the basic functions $Z(\Gamma \,|\, T)$ entered the *original* function $F(T)$ with the coefficients 1 and ϵ. If instead we write

$$F(T) = a{-\!\bullet\!-} + \epsilon b{-\!\bullet\!-\!\bullet\!-} = aT + \epsilon b T^2, \qquad \text{i.e.,} \qquad \widehat{F}({-\!\bullet\!-}) = a, \qquad \widehat{F}({-\!\bullet\!-\!\bullet\!-}) = \epsilon b, \tag{13}$$

then the coefficient a appears in the left-hand sides of (4):

$$aQ({-\!\bullet\!-}) = -\mathcal{P}_-\big\{F({-\!\bullet\!-})\big\} = 0,$$

$$aQ({-\!\bullet\!-\!\bullet\!-}) = -\mathcal{P}_-\big\{F({-\!\bullet\!-\!\bullet\!-})\big\} = -\epsilon b,$$

$$aQ({-\!\bullet\!-\!\bullet\!-\!\bullet\!-}) = -2\mathcal{P}_-\big\{F({-\!\bullet\!-\!\bullet\!-})Q({-\!\bullet\!-\!\bullet\!-})\big\} = \frac{2\epsilon^2 b^2}{a}.$$

Hence,

$$\widetilde{T} = T + Q({-\!\bullet\!-})T + Q({-\!\bullet\!-\!\bullet\!-})T^2 + Q({-\!\bullet\!-\!\bullet\!-\!\bullet\!-})T^3 + \cdots = T - \frac{\epsilon b T^2}{a} + \frac{2\epsilon^2 b^2 T^3}{a^2} - \cdots$$

and

$$F(\widetilde{T}) = a\widetilde{T} + \epsilon b \widetilde{T}^2 = a\left(T - \frac{\epsilon b T^2}{a} + \frac{2\epsilon^2 b^2 T^3}{a^2} - \cdots\right) + \epsilon b\left(T - \frac{\epsilon b T^2}{a} + \frac{2\epsilon^2 b^2 T^3}{a^2} - \cdots\right)^2 = aT. \tag{14}$$

I.1.3. Applying the second-level forest formula. Second-level forest formula (5) produces the renormalized function $F(\widetilde{T})$ directly. We already know what it is from (14), and this answer is very simple:

$$F(\widetilde{T}) = F_{\mathrm{R}}(T) = aT.$$

This means that in this case the only nonzero output of (5) should be $F_{\mathrm{R}}({-\!\bullet\!-}) = a$ and that $F_{\mathrm{R}}(\Gamma) = 0$ for all other graphs. The first statement is immediately reproduced by (5): there is only one term in the right-hand side of (5). For more complicated Γ, exact cancellation should occur between different contributions to (5).

We demonstrate this for ${-\!\bullet\!-\!\bullet\!-}$, where the two trees

$${-\!\bullet\!-\!\bullet\!-}, \qquad {-\!\boxed{\bullet\!-\!\bullet}\!-}$$

can be built. We do not consider trees that have box subgraphs with one vertex inside, like ${-\!\boxed{\bullet}\!-}$, because $\mathcal{P}_- F({-\!\bullet\!-}) = 0$. We also do not draw the external box that corresponds to the whole graph. Therefore,

$$F_{\mathrm{R}}({-\!\bullet\!-\!\bullet\!-}) = F({-\!\bullet\!-\!\bullet\!-}) + F({-\!\bullet\!-})\frac{1}{F({-\!\bullet\!-})}(-\mathcal{P}_-)F({-\!\bullet\!-\!\bullet\!-}) = F({-\!\bullet\!-\!\bullet\!-}) - \mathcal{P}_- F({-\!\bullet\!-\!\bullet\!-}) = 0. \tag{15}$$

The graph ${-\!\bullet\!-\!\bullet\!-\!\bullet\!-}$ has the following set of embedded box subgraphs (only subtrees with a nonzero contribution to (5)) are shown): ${-\!\boxed{\bullet\!-\!\bullet}\!-\!\bullet\!-}$, $\boxed{-\!\boxed{\bullet\!-\!\bullet}\!-\!\bullet\!-}$ and ${-\!\bullet\!-\!\boxed{\bullet\!-\!\bullet}\!-}$, $\boxed{-\!\bullet\!-\!\boxed{\bullet\!-\!\bullet}\!-}$. It is quite obvious that the first two and the second two trees contribute the same, and we hence take only the first two graphs into account and double the result:

$$F_{\mathrm{R}}({-\!\bullet\!-\!\bullet\!-\!\bullet\!-}) = 2\bigg\{F({-\!\bullet\!-\!\bullet\!-})\frac{1}{F({-\!\bullet\!-})}(-\mathcal{P}_-)F({-\!\bullet\!-\!\bullet\!-}) +$$

$$+ F({-\!\bullet\!-})\frac{1}{F({-\!\bullet\!-})}(-\mathcal{P}_-)\left[F({-\!\bullet\!-\!\bullet\!-})\frac{1}{F({-\!\bullet\!-})}(-\mathcal{P}_-)F({-\!\bullet\!-\!\bullet\!-})\right]\bigg\}. \tag{16}$$

Hence, $F_R(\text{-•-•-•-})$ vanishes,

$$F_{\mathrm{R}}(\text{-•-•-•-}) = -\frac{2}{F(\text{-•-})}\{F(\text{-•-•-})\mathcal{P}_- F(\text{-•-•-}) - \mathcal{P}_- F(\text{-•-•-})\mathcal{P}_- F(\text{-•-•-})\} = 0,$$

because $F(\text{-•-•-})$ is $\mathcal{P}$-negative.

It is now easy to understand how cancellations occur: all trees for some particular graph can be divided into pairs: the only difference between the trees in one pair is that one of them has an "external" box, i.e., $\gamma_0 = \gamma_1 = \Gamma_1$ in terms of (5). Because there is the additional projection operator $(-\mathcal{P}_-)$, the two graphs in each pair mutually cancel.

In what follows, we do not give examples of applying the second-level forest formula in cases where it leads to an obvious result, because the same argument can be used there to prove the contributions of the majority of graphs cancel.

I.1.4. General projection operator. As the next generalization, we do not fix the specific ϵ dependence of the coefficients $A(\epsilon)$ and $B(\epsilon)$ in

$$F(T) = A(\epsilon)T + B(\epsilon)T^2 = A(\epsilon)\text{-•-} + B(\epsilon)\text{-•-•-},$$

i.e.,

$$\widehat{F}(\text{-•-}) = A(\epsilon), \qquad \widehat{F}(\text{-•-•-}) = B(\epsilon) \tag{17}$$

(e.g., they can be arbitrary Laurent series), and also do not fix the specific choice of the projection operator $\mathcal{P}_-$ (e.g., it can select terms with ϵ^k and $k > k_0$ or terms with $\epsilon^k \sinh\epsilon$). Equation (4) works in all situations! Indeed, the counterterms are

$$A(\epsilon)Q(\text{-•-}) = -\mathcal{P}_-\big\{F(\text{-•-})\big\} = -A_-,$$

i.e.,

$$Q(\text{-•-}) = -\frac{A_-}{A},$$

$$A(\epsilon)Q(\text{-•-•-}) = -\mathcal{P}_-\big\{F(\text{-•-•-})\big(1 + 2Q(\text{-•-}) + [Q(\text{-•-})]^2\big)\big\} =$$

$$= -\left[B\left(1 - \frac{A_-}{A}\right)^2\right]_- = -\left(\frac{BA_+^2}{A^2}\right)_-,$$

$$Q[3] = Q(\text{-•-•-•-}) = -\frac{1}{A}\mathcal{P}_-\{F[2](2Q[2] + 2Q[2]Q[1])\} =$$

$$= \frac{2}{A}\left(B\frac{1}{A}\left(\frac{BA_+^2}{A^2}\right)_-\left\{1 - \frac{A_-}{A}\right\}\right)_- = \frac{2}{A}\left(\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right)_-,$$

$$Q[4] = -\frac{1}{A}\mathcal{P}_-(F[2]Q^2[2] + 2F[2]Q[3] + 2F[2]Q[3]Q[1]) =$$

$$= -\frac{1}{A}\left(\frac{B}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-^2 + \frac{4BA_+}{A^2}\left(\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right)_-\right)_-, \qquad \ldots.$$

Therefore,

$$\widetilde{T} = A_+\frac{T}{A} - \left(\frac{BA_+^2}{A^2}\right)_-\frac{T^2}{A} + 2\left(\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right)_-\frac{T^3}{A} -$$

$$- \left\{4\left(\frac{BA_+}{A^2}\left(\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right)_-\right)_- + \left(\frac{B}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-^2\right)_-\right\}\frac{T^4}{A} + \cdots$$

and

$$
\begin{aligned}
F(\widetilde{T}) = A\widetilde{T} + B\widetilde{T}^2 = A_+T + \left(\frac{BA_+^2}{A^2} - \left(\frac{BA_+^2}{A^2}\right)_-\right)T^2 - {}\\
-2\left[\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_- - \left(\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right)_-\right]T^3 + {}\\
+\left\{4\frac{BA_+}{A^2}\left(\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right)_- - 4\left(\frac{BA_+}{A^2}\left(\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right)_-\right)_- + {}\right.\\
\left. + \frac{B}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-^2 - \left(\frac{B}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-^2\right)_-\right\}T^4 + \dots.
\end{aligned}
$$

Clearly, all terms before each power of T combine into a $\mathcal{P}$-positive expression:

$$
\begin{aligned}
F(\widetilde{T}) = A_+T + \left(\frac{BA_+^2}{A^2}\right)_+ T^2 - 2\left[\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right]_+ T^3 + {}\\
+\left[4\frac{BA_+}{A^2}\left(\frac{BA_+}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-\right)_- + \frac{B}{A^2}\left(\frac{BA_+^2}{A^2}\right)_-^2\right]_+ T^4 + \dots.
\end{aligned}
$$

The right-hand side contains only pure + projections, but we note that in this general case, the renormalized $F(\widetilde{T})$ is no longer quadratic in T, in contrast to the original $F(T)$.

I.1.4.1. Concrete example. To the clarify last statement, we consider a simple example. Let $\mathcal{P}_-$ select only terms of the order ϵ, $\mathcal{P}_-F = (\partial F/\partial\epsilon)_{\epsilon=0}$, and let

$$F(T) = T + b\epsilon T^2 + c\epsilon^2 T^3.$$

For such $F(T)$, we can easily verify that there is only one nonzero Q, namely, $Q(-\!\bullet\!-\!\bullet\!-) = -b\epsilon$. It follows that $\widetilde{T} = T - b\epsilon T^2$. Hence, the renormalized $F(T)$ is no longer quadratic in T:

$$
\begin{aligned}
F(T) = F\big(\widetilde{T}(T)\big) = T - b\epsilon T^2 + b\epsilon(T - b\epsilon T^2)^2 + c\epsilon^2(T - b\epsilon T^2)^3 = {}\\
= T + (c - 2b^2)\epsilon^2T^3 + (b^3 - 3cb)\epsilon^3T^4 + 3cb^2\epsilon^4T^5 - cb^3\epsilon^5T^6. \qquad (18)
\end{aligned}
$$

This, so it seems, contradicts all the previous examples: we should expect $F(\widetilde{T}) = T + c\epsilon^2T^3$. But the point is that the Q that would renormalize F in such a way would contain $\mathcal{P}$-positive terms and therefore cannot arise from the forest formula. The forest formula provides $Q(\Gamma)$ satisfying constraint (2).

This is the first example demonstrating that the renormalized F can be very different from the original function, more sophisticated or simplified. This implies that second-level forest formula (5), directly describing this modification, is not quite trivial. We show how (18) follows from the second-level forest formula.

For $F(-\!\bullet\!-\!\bullet\!-)$, we obtain the same result as in (15). For $F(-\!\bullet\!-\!\bullet\!-\!\bullet\!-)$, we have (cf. (16))

$$
\begin{aligned}
F_{\mathrm{R}}(-\!\bullet\!-\!\bullet\!-\!\bullet\!-) = F(-\!\bullet\!-\!\bullet\!-\!\bullet\!-) + {}\\
+2\left\{F(-\!\bullet\!-\!\bullet\!-)\frac{1}{F(-\!\bullet\!-)}(-\mathcal{P}_-)F(-\!\bullet\!-\!\bullet\!-) + {}\right.\\
\left. + F(-\!\bullet\!-)\frac{1}{F(-\!\bullet\!-)}(-\mathcal{P}_-)\left[F(-\!\bullet\!-\!\bullet\!-)\frac{1}{F(-\!\bullet\!-)}(-\mathcal{P}_-)F(-\!\bullet\!-\!\bullet\!-)\right]\right\} = {}\\
= c\epsilon^2 + 2\{b\epsilon(-\mathcal{P}_-)b\epsilon + (-\mathcal{P}_-)[b\epsilon(-\mathcal{P}_-)b\epsilon]\},
\end{aligned}
$$

where $(-\mathcal{P}_-)\big[b\epsilon(-\mathcal{P}_-)b\epsilon\big] = 0$, and we obtain $F_{\mathrm{R}}(\text{—•—•—•—}) = c\epsilon^2 - 2b^2\epsilon^2$. We can similarly reproduce the coefficients of T^4, T^5, and T^6. The interesting point is to understand why $F_{\mathrm{R}}(\Gamma)$ vanishes for chain graphs with more than six vertices, as predicted by (18). This is also almost obvious: it is easy to see that trees with embedded box graphs are not allowed, because $\mathcal{P}_-$ in (5) acts upward along the branches of the tree and $\mathcal{P}_-\epsilon^n = 0$ for $n > 1$. All our boxes can contain only two vertices, and the only nonzero $F(\Gamma)$ are those for chain graphs with one, two, or three vertices. Multiplying two by three gives six, and $F_{\mathrm{R}}(\Gamma)$ vanishes for all the chain graphs with more than six vertices, which corresponds to (18).

I.1.4.2. Another example. To further deepen our understanding of problems that are solved using the forest formula, we consider one more example with the same projection operator as in Sec. I.1.4.1 but with a different choice of the function,

$$F(T) = \frac{1}{\epsilon}T + b\epsilon T^2.$$

In this case, there is a single nonzero counterterm $Q(\Gamma)$:

$$\frac{1}{\epsilon}Q(\text{—•—•—}) = -\mathcal{P}_-\big\{F(\text{—•—•—})\big\} = -\mathcal{P}_-\{b\epsilon\} = -b\epsilon, \qquad Q(\text{—•—•—}) = -b\epsilon^2.$$

We see that $\mathcal{P}_+Q(\text{—•—•—}) = -b\epsilon^2 \neq 0$, which contradicts condition (2). This is an example where an appropriate modification of (2), mentioned in footnote 1, is unavoidable: $F(\text{—•—})Q(\text{—•—•—})$ rather than $Q(\text{—•—•—})$ itself should be $\mathcal{P}$-negative. This is indeed the case: $\mathcal{P}_+F(\text{—•—})Q(\text{—•—•—}) = -\mathcal{P}_+b\epsilon = 0$.

I.1.5. Loop diagrams, a need for vacuum energy: $\boldsymbol{F(T) \to F(v,T) = v + T + \epsilon T^2}$. We now represent T and T^2 by two loop diagrams ○• and •○•. But for both these diagrams $\Gamma/\Gamma = \bullet$ is a vertex of valence zero (no external legs), which is not represented in our function $F(T)$. To make formula (4) applicable in this situation, we must supplement F with an additional term associated with this diagram. If we let v denote the corresponding coupling constant, then our modified function can be written as

$$F(v,T) = v + T + \epsilon T^2 = \bullet + \text{○•} + \epsilon\,\text{•○•}. \tag{19}$$

We can now apply (4), and the only nonzero $Q(\Gamma)$ is

$$Q(\text{•○•}) = -\mathcal{P}_-\{F(\text{•○•})\} = -\epsilon. \tag{20}$$

Because corrections to the v and T respectively correspond to the graphs with zero and two external legs, (20) provides only a shift of v:

$$\tilde{v} = v + Q(\text{•○•}) = v - \epsilon T^2, \qquad \widetilde{T} = T, \tag{21}$$

and the renormalized $F(\tilde{v},\widetilde{T}) = \tilde{v} + T + \epsilon T^2 = v + T$ is therefore indeed independent of ϵ.

I.1.6. Mixed chain-loop diagrams and restoration of tensor structures. We now represent T by the loop ○• but leave T^2 represented by the chain diagram —•—•—. This time we have two coupling constants and

$$F(v,T) = \alpha v + (a+b)T + \epsilon c T^2 = \alpha\bullet + a\text{—•—} + b\text{○•} + \epsilon c\text{—•—•—}. \tag{22}$$

From (4), we now obtain

$$aQ(-\bullet-\bullet-) = -\mathcal{P}_-\big\{F(-\bullet-\bullet-)\big\} = -\epsilon c, \tag{23}$$

$$\alpha Q(\bullet\!\bigcirc\!\bullet) = -\mathcal{P}_-\big\{F(\bigcirc\!\!\bullet)Q(-\bullet-\bullet-)\big\} = -b\left(-\frac{\epsilon c}{\alpha}\right) = \frac{\epsilon bc}{\alpha}, \tag{24}$$

$$aQ(-\bullet-\bullet-\bullet-) = -\mathcal{P}_-\big\{2Q(-\bullet-\bullet-)F(-\bullet-\bullet-)\big\} = 2\frac{(\epsilon c)^2}{a}, \tag{25}$$

$$\alpha Q(\bullet\!\overset{\bullet}{\bigcirc}\!\bullet) = -\mathcal{P}_-\{F(\bigcirc\!\!\bullet)Q(-\bullet-\bullet-\bullet-)\} = -\frac{b}{\alpha}\mathcal{P}_-\{Q(-\bullet-\bullet-\bullet-)\} = -2\frac{(\epsilon c)^2}{a}\frac{b}{\alpha}. \tag{26}$$

To understand why the coefficients 2 and 3 (the numbers of ways we can cut the corresponding chain graph from the loop one) do not appear in (24) and (26), we should restore the tensor structure of the vertices v and T. Our problem then takes the form

$$\alpha\tilde{v} + a\widetilde{T}^{ij} + b\widetilde{T}^{ll} + \epsilon c\widetilde{T}^{il}\widetilde{T}^{lj} = \alpha v + aT^{ij} + bT^{ll}.$$

Because there are two different tensor structures, this relation is not one equation but a system:

$$a\widetilde{T}^{ij} + \epsilon c\widetilde{T}^{il}\widetilde{T}^{lj} = aT^{ij},$$

$$\alpha\tilde{v} + b\widetilde{T}^{ll} = \alpha v.$$

Hence, $\alpha\tilde{v} = \alpha v - b\widetilde{T}^{ll}$, and the loop graphs $\bullet\!\bigcirc\!\bullet$, $\bullet\!\overset{\bullet}{\bigcirc}\!\bullet$, ... are just the corresponding chain graphs $-\bullet-\bullet-$, $-\bullet-\bullet-\bullet-$, ... with their ends glued together.

For the renormalized v and T, we obtain

$$\begin{aligned}\tilde{v} &= \bullet + Q(\bullet\!\bigcirc\!\bullet)\,\bullet\!\bigcirc\!\bullet + Q(\bullet\!\overset{\bullet}{\bigcirc}\!\bullet)\,\bullet\!\overset{\bullet}{\bigcirc}\!\bullet + \cdots = v + \frac{\epsilon cb}{a\alpha}T^2 - 2\frac{(\epsilon c)^2 b}{a\alpha}T^3 + \ldots,\\ \widetilde{T} &= -\bullet- + Q(-\bullet-\bullet-)-\bullet-\bullet- + Q(-\bullet-\bullet-\bullet-)-\bullet-\bullet-\bullet- + \cdots = T - \frac{\epsilon c}{a}T^2 + 2\left(\frac{\epsilon c}{a}\right)^2 T^3 - \ldots.\end{aligned} \tag{27}$$

Because the values of $Q\big[n^{(\mathrm{loop})}\big]$ for the loop diagrams are determined by the values of $Q[n]$ for the corresponding chain diagrams, we can sum using a generating function, the same as in (11):

$$F(\tilde{v},\widetilde{T}) = \alpha\tilde{v} + (a+b)\widetilde{T} + \epsilon c\widetilde{T}^2 =$$

$$= \alpha\left(v + \frac{\epsilon bc}{a\alpha}T^2 + \cdots\right) + (a+b)\left(T - \frac{\epsilon c}{a}T^2 + \cdots\right) + \epsilon c\left(T - \frac{\epsilon c}{a}T^2 + \cdots\right)^2 = \alpha v + (a+b)T.$$

We note that even if α was originally set to zero, it acquires an α-independent correction in the renormalization process. The vertex of valence zero is automatically generated by the forest formula. But this happens in a singular way: the α-dependent correction is first divided and then multiplied by $\alpha \to 0$.

I.2. General polynomial from the bivalent vertex

We are now ready to attack a more general problem than in the preceding sections. For a function $F(T)$, we take a sum of two polynomials with the same projection operator as in (7),

$$F(T) = R(T) + \epsilon G(T).$$

The renormalization problem reduces to the equation

$$R(\widetilde{T}) + \epsilon G(\widetilde{T}) = R(T),$$

and the solution is represented as a power series in ϵ,

$$\widetilde{T} = T - \epsilon\frac{G(T)}{R'(T)} + \epsilon^2\frac{G(T)}{2\big(R'(T)\big)^3}\big(2R'(T)G'(T) - R''(T)G(T)\big) + \dots. \tag{28}$$

The obvious question is what is the origin of the various powers of $R'(T)$ from the standpoint of the forest formula.

I.2.1. The simplest nontrivial $F(T) = T + \alpha T^2 + \epsilon T^3$. We begin with the simplest nontrivial function R. Let $R(T) = T + \alpha T^2$ and $G(T) = \epsilon T^3$. Substituting this in (28), we obtain

$$\widetilde{T} = T - \epsilon\frac{T^3}{1 + 2\alpha T} + \epsilon^2\frac{T^5(3 + 5\alpha T)}{(1 + 2\alpha T)^3} + \dots. \tag{29}$$

Expanding the denominators in αT, we obtain

$$\widetilde{T} = T - \epsilon(T^3 - 2\alpha T^4 + 4\alpha^2T^5 - 8\alpha^3T^6 + \cdots) + \epsilon^2(3T^5 - 13\alpha T^6 + 42\alpha^2T^7 + \cdots) + \dots. \tag{30}$$

We now attack this problem with the forest formula:

$$F(T) = T + \alpha T^2 + \epsilon T^3 = -\!\bullet\!- + \alpha -\!\bullet\!-\!\bullet\!- + \epsilon -\!\bullet\!-\!\bullet\!-\!\bullet\!-,$$

$$Q[3] = Q(-\!\bullet\!-\!\bullet\!-\!\bullet\!-) = -\epsilon,$$

$$Q[4] = -\mathcal{P}_-(2F[2]Q[3]) = 2\alpha\epsilon,$$

$$Q[5] = -\mathcal{P}_-(2F[2]Q[4] + 3F[3]Q[3]) = 3\epsilon^2 - 4\alpha^2\epsilon,$$

$$Q[6] = -\mathcal{P}_-(F[2]Q^2[3] + 2F[2]Q[5] + 3F[3]Q[4]) = 8\alpha^3\epsilon - 13\alpha\epsilon^2.$$

Hence, for $\widetilde{T}$, we have

$$\widetilde{T} = T - \epsilon T^3 + 2\alpha\epsilon T^4 + (3\epsilon^2 - 4\alpha^2\epsilon)T^5 + (8\alpha^3\epsilon - 13\alpha\epsilon^2)T^6 + \cdots =$$

$$= T - \epsilon(T^3 - 2\alpha T^4 + 4\alpha^2T^5 - 8\alpha^3T^6 + \cdots) + \epsilon^2(3T^5 - 13\alpha T^6 + \cdots) + \dots,$$

i.e., we have (30) but expanded first in αT and then in ϵ.

It is instructive to explicitly sum all the terms contributing to the first and second orders in ϵ. As usual, this can be done using the generating functions $Q_{(1)}(t) = \sum_{n=3} t^nQ_{(1)}[n]$ and $Q_{(2)}(t) = \sum_{n=3} t^nQ_{(2)}[n]$, where $Q_{(1)}[n]$ and $Q_{(2)}[n]$ denote the respective contributions to $Q[n]$ of the orders ϵ and ϵ^2.

Because $Q[n]$ does not contain terms of the order ϵ^0, only the terms with a single box subgraph contribute to $Q_{(1)}[n+1]$:

$$Q_{(1)}[n+1] = -\mathcal{P}_-(2F[2]Q_{(1)}[n]).$$

For $Q_{(1)}(t)$, this yields

$$Q_{(1)}(t) = Q[3]t^3 - 2\alpha tQ_{(1)}(t), \qquad Q_{(1)}(t) = \frac{t^3Q[3]}{1 + 2\alpha t} = -\epsilon\frac{t^3}{1 + 2\alpha t}.$$

For $Q_{(2)}[n+1]$, we should take three contributions into account:

$$F[2]Q_2[n], \qquad F[3]Q_{(1)}[n], \qquad F[2]Q_{(1)}[n]Q_{(1)}[k].$$

We obtain

$$Q_{(2)}[n+3] = -\mathcal{P}_-\left(2F[2]Q_{(2)}[n+2] + 3F[3]Q_{(1)}[n+1] + F[2]\sum_{k=3}^{n} Q_{(1)}[k]Q_{(1)}[n+3-k]\right).$$

Therefore, $Q_{(2)}(t)$ satisfies

$$Q_{(2)}(t) = -\big(2tF[2]Q_{(2)}(t) + 3t^2F[3]Q_{(1)}(t) + F[2]Q_{(1)}^2(t)\big),$$

and

$$Q_{(2)}(t) = -\frac{3\epsilon t^2 Q_{(1)}(t) + \alpha Q_{(1)}^2(t)}{1+2\alpha t} = \frac{\epsilon^2 t^5(3+5\alpha t)}{(1+2\alpha t)^3}.$$

Clearly, we reproduce (29) after resumming.

I.2.2. General $F(T)$. To understand how the forest formula can differentiate polynomials and provide $R'(T)$ in the denominator in (28), we must sum the terms of the order ϵ^1. Exactly as in the preceding subsection, we do this by solving an equation for the generating function

$$F(T) = T + \sum_{n=2}^{R} r_n T^n + \epsilon\left(\sum_{n=1}^{G} g_n T^n\right).$$

Here, R and G denote degrees of the respective polynomials $R(T)$ and $G(T)$, and $r_1 = 1$ for simplicity. If

$$Q_{(1)}(t) = \sum_{n=1} t^n Q_{(1)}[n],$$

then

$$\begin{aligned} Q_{(1)}[n] = \; &-\mathcal{P}_-^{(1)}\{2F[2]Q_{(1)}[n-1] + 3F[3]Q_{(1)}[n-2] + \\ &+ 4F[4]Q_{(1)}[n-3] + \cdots + RF[R]Q_{(1)}[n-R+1]\} \end{aligned}$$

for $n \geq R$, where $\mathcal{P}_-^{(1)}$ is a projection operator that selects only the ϵ^1 terms. The coefficients $2, 3, \ldots, R$ count the numbers of ways that we can cut a box subgraph of the corresponding length from the chain graph $[n]$. Expressed in terms of $Q_{(1)}(t)$, this recurrent relation becomes

$$Q_{(1)}(t) = -(2r_2 t + 3r_3 t^2 + \cdots + Rr_{\mathrm{R}} t^{R-1})Q_{(1)}(t) - (g_1 t - g_2 t^2 - \cdots - g_G t^G)$$

and finally

$$Q_{(1)}(t) = -\frac{G(t)}{R'(t)}.$$

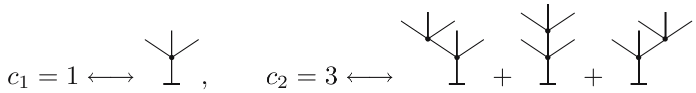
We can similarly sum term by term for all orders in ϵ.

I.3. Cubic equation: Trivalent vertex and loops

Let T represent a trivalent vertex. We can then use the forest formula to solve the cubic equation $\widetilde{T} + \epsilon\widetilde{T}^3 = T$, assuming that the projection operator is given by (7). The solution is

$$\widetilde{T} = T - \epsilon T^3 + 3\epsilon^2 T^5 - 12\epsilon^3 T^7 + 55\epsilon^4 T^9 + \cdots = T + \sum_{n=1}(-1)^n c_n \epsilon^n T^{2n+1},$$

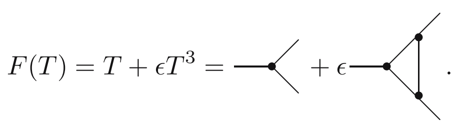
where c_n are the numbers of rooted trivalent trees with exactly n vertices, i.e.,



(31)

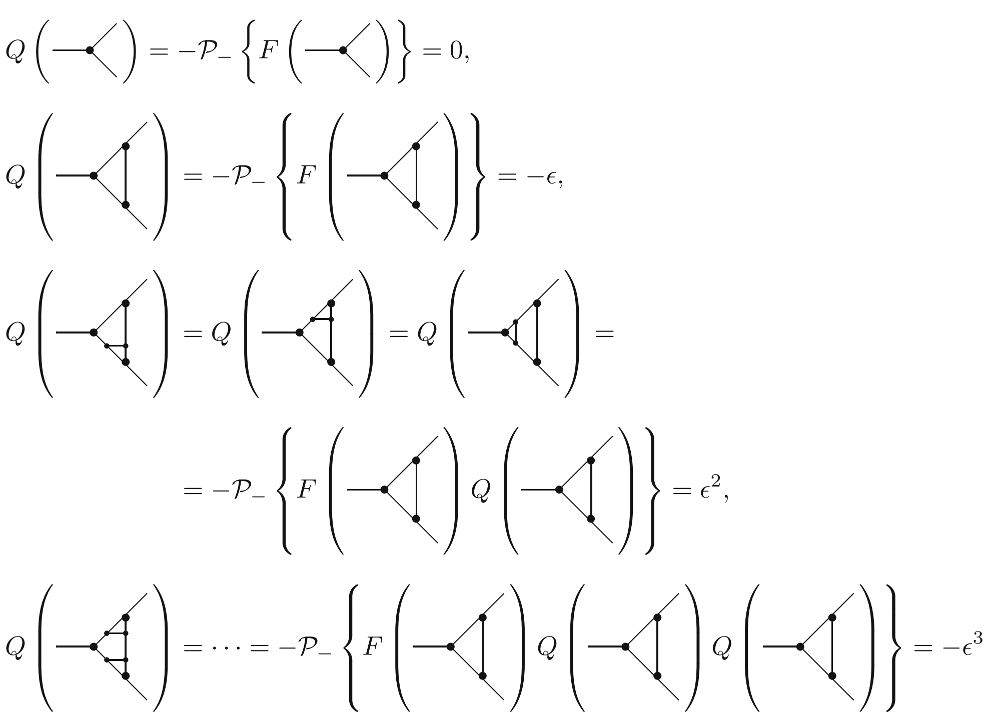
We obtain this solution from the forest formula. As in the case of a bivalent vertex, there are different possibilities to represent the function $F(T)$ and thus different realizations of the renormalization procedure.

I.3.1. First realization. We can represent $T + \epsilon T^3$, for example, as



(32)

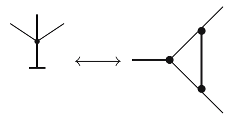
When we start calculating $Q(\Gamma)$, the forest formula itself selects the relevant graphs from the huge set of all possible graphs with trivalent vertices. We have

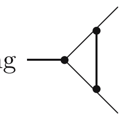


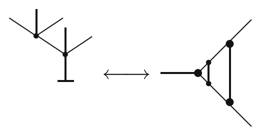
It is easy to see that because embedded graphs do not contribute to the first-level forest formula, all nonzero $Q(\Gamma)$ have the form

$$Q(\Gamma) = (-1)^{(k-1)/2}\epsilon^{(k-1)/2},$$

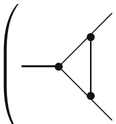
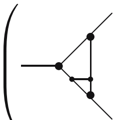
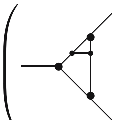
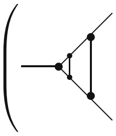
where k is the number of vertices in Γ. Therefore, the problem reduces to counting the number of graphs of order k that give a nonzero contribution. But this becomes a simple task if we observe a one-to-one correspondence between “triangle” graphs and rooted trees from (31):

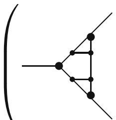


When we add three edges to some vertex of the original tree, this corresponds to inserting  at the corresponding vertex of our graph, for example,

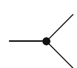
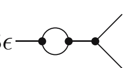


Therefore, we obtain the following result for $\widetilde{T}$ from the forest formula:

$\widetilde{T} = T + Q\Bigl($  $\Bigr)T^3 + \Bigl\{Q\Bigl($  $\Bigr) + Q\Bigl($  $\Bigr) + Q\Bigl($  $\Bigr)\Bigr\}T^5 +$

$+\Bigl\{Q\Bigl($  $\Bigr) + \cdots\Bigr\}T^7 + \cdots = T + \sum_{n=1}(-1)^n c_n \epsilon^n T^{2n+1}.$

I.3.2. Second realization. We can represent the same function $F(T)$ differently:

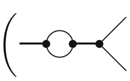
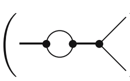
$F(T) = T + \epsilon T^3 =$  $+\,3\epsilon$  . (33)

Here we insert the coefficient 3 reflecting the problem symmetry. It becomes obvious if we restore the indices of a triple totally symmetric vertex $T \to T_{ijk}$ such that (sums over repeated indices are implied)

$$F_{ijk}(T) = T_{ijk} + \epsilon\bigl((TT)_{im}T_{mjk} + (TT)_{jm}T_{imk} + (TT)_{km}T_{ijm}\bigr), \qquad (TT)_{im} = T_{ipq}T_{mpq}, \tag{34}$$

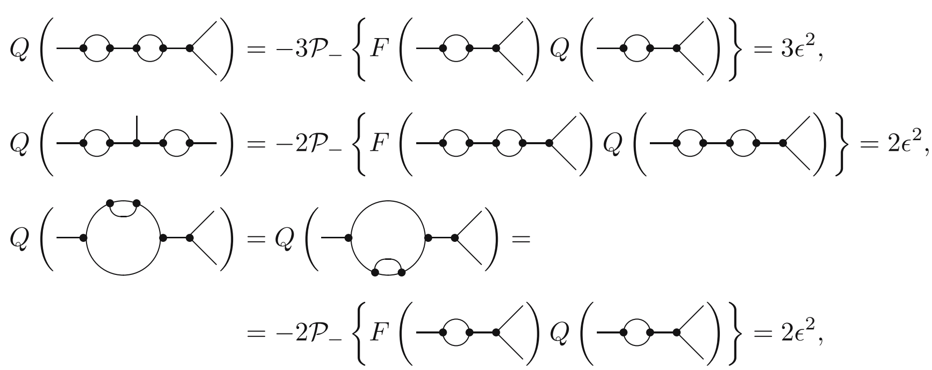
and there are in fact three different one-loop diagrams.

The first counterterms are

$Q\Bigl($  $\Bigr) = -\mathcal{P}_-\Bigl\{F\Bigl($  $\Bigr)\Bigr\} = -\epsilon.$ (35)

We note that there is exactly the same diagram in the left-hand and right-hand sides; hence, the coefficient 3 does not arise in this formula.

There are diagrams with three different topologies at the next level:



(36)

(37)

(38)

where the coefficients 3 in (36) and 2 in (37) and (38) correspond to the number of different box subgraphs.

Taking three different orientations of the diagrams into account, we obtain the counterterm $Q(T)$:

$$\widetilde{T} = T - 3\epsilon T^3 + 3 \cdot (3 + 2 + 2 \times 2)\epsilon^2 T^5 + \cdots$$

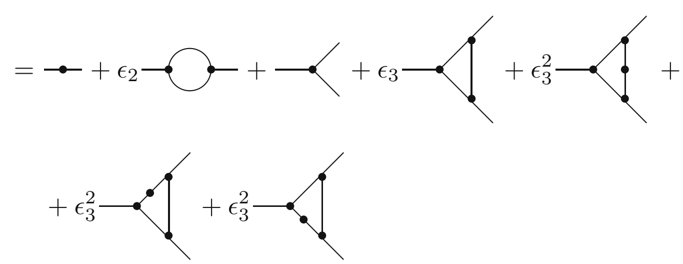
and

$$F(\widetilde{T}) = \widetilde{T} + 3\epsilon\widetilde{T}^3 = (T - 3\epsilon T^3 + 27\epsilon^2 T^5 + \cdots) + 3\epsilon(T - 3\epsilon T^3 + 27\epsilon^2 T^5 + \cdots)^3 = T.$$

I.4. Cubic equation: Two vertices, bivalent and trivalent

We already encountered a problem with more than one coupling constant, but our example was rather simple. We now try to show a specific feature of the forest formula when working with several coupling constants. We consider

$$F(T) = T_2 + \epsilon_2 T_3^2 + T_3 + \epsilon_3 T_3^3 + 3\epsilon_3^2 T_3^3 T_2 =$$



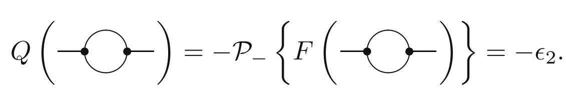
(39)

and let $\mathcal{P}_-$ select all terms $\epsilon_2^\alpha \epsilon_3^\beta$ with $|\alpha| + |\beta| \geq 1$.

Again, as in Sec. I.1.6, because there are different tensor structures, the equation $F(\widetilde{T}) = T_2 + T_3$ implies renormalization of two different functions:

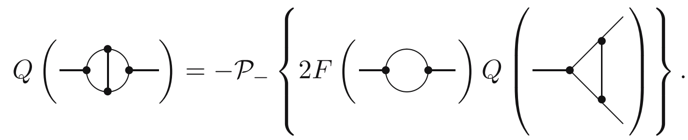
$$\widetilde{T}_2 + \epsilon_2 \widetilde{T}_3^2 = T_2, \qquad \widetilde{T}_3 + \epsilon_3 \widetilde{T}_3^3 + 3\epsilon_3^2 \widetilde{T}_3^3 \widetilde{T}_2 = T_3.$$

But the two equations are interrelated this time, and as a result of the iteration procedure, we should find $\widetilde{T}_2$ and $\widetilde{T}_3$ simultaneously. We try to solve this problem using the forest formula. Everything is simple for the first graph:

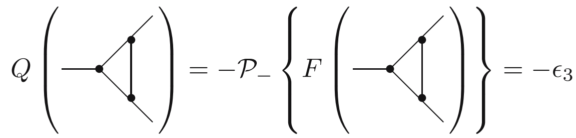


(40)

But when we examine the next graph with two external legs, a difficulty arises: it is unknown how to find

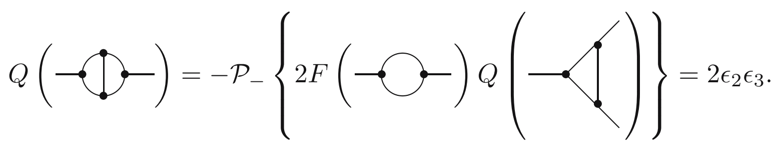


It turns out that we must first calculate



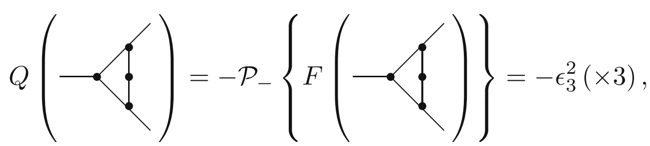
(41)

and only then return to

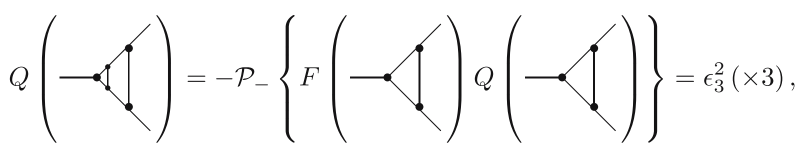


(42)

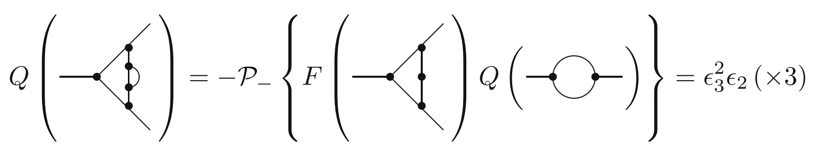
For the next graphs, we have



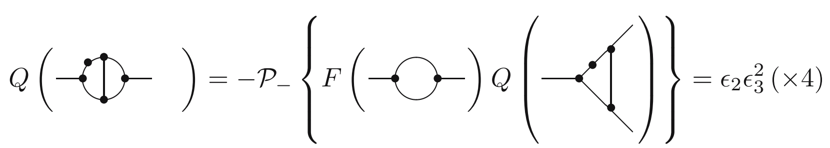
(43)



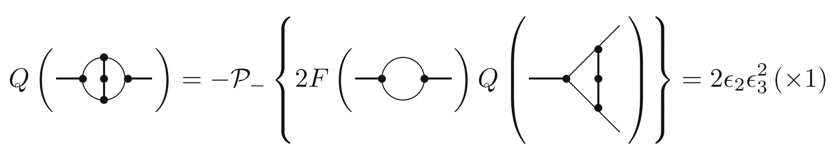
(44)



(45)



(46)



(47)

where $(\times 3)$, $(\times 4)$, ... denote the numbers of possible positions of the dot or, in the case of (44), the number of orientations of the corresponding graph. We again note that we need (40) for calculating (45) and we need (43) for (46) and (47), although these graphs have different numbers of external legs.

The forest formula itself selects the graphs that contribute to the renormalization. Because of the connection between the renormalization processes for T_2 and T_3, we cannot determine Q first for graphs with two external legs and then for those with three external legs; this should be done simultaneously order by order:

$$\widetilde{T}_2 = T_2 - \epsilon_2 \overset{(40)}{T_3^2} + 2\epsilon_2\epsilon_3 \overset{(42)}{T_3^4} + \overset{(46)}{4\epsilon_2\epsilon_3^2 T_2 T_3^4} + \overset{(47)}{2\epsilon_2\epsilon_3^2 T_2 T_3^4} + \dots, \tag{48}$$

$$\widetilde{T}_3 = T_3 - \epsilon_3 \overset{(41)}{T_3^3} - \overset{(43)}{3\epsilon_3^2 T_2 T_3^3} + \overset{(44)}{3\epsilon_3^2 T_3^5} + \overset{(45)}{3\epsilon_2\epsilon_3^2 T_3^5} + \dots. \tag{49}$$

To make the origin of every term clear, the number of the formula corresponding to the contribution is written above each term. It is easy to verify that (48) and (49) solve the original problem.

II. QFT examples

It is rather natural to apply forest formulas in QFT; this is what they were originally designed for. Perturbation theories in QFT often suffer from divergences originating from momentum integrations in loop diagrams, i.e., from sums over tensor indices in our diagram expansions. To give a meaning to the divergent integral, regularizations are applied, which introduces a dependence on additional parameters like the ultraviolet cut-off Λ or $\epsilon = 4 - d$ in dimensional regularization or masses M_i in the Pauli–Villars approach. These parameters must be eliminated at the end of the calculation. This cannot be done by simply removing them from the answer because this would change relations between different correlators, i.e., the shape of the partition function. After an arbitrary change, this partition function and these correlators cannot be obtained from any kind of functional integral: they thus fail to represent any kind of quantum mechanical system. This cannot occur if we do not change the shape of the partition function but modify its arguments, i.e., the bare coupling constants, which are exactly our parameters $\widetilde{T}$ in the preceding sections. The forest formula gives an expression for these bare coupling constants $\widetilde{T}$ in terms of the physical constants T:

$$\widetilde{T} \equiv T + Q(T),$$

where $Q(T)$ are singular counterterms that make the partition function finite. In this sense, the forest formula resolves the renormalization problem in QFT.

The possibility of applying the forest formula is independent of the type and origin of the function $F(T)$ that we want to renormalize. The only thing we must consider for performing a self-consistent renormalization is the validity of the vertex criterion: for every graph Γ with a nonzero $Q(\Gamma)$, there should be a vertex Γ/Γ in $F(T)$, i.e., $F(\Gamma/\Gamma) \neq 0$ if $Q(\Gamma) \neq 0$. If this criterion is satisfied, then the renormalization in the QFT can be performed at the level of the partition function, i.e., for all the correlators simultaneously. But it is often possible to restrict the consideration to correlators with a particular type of external structure.

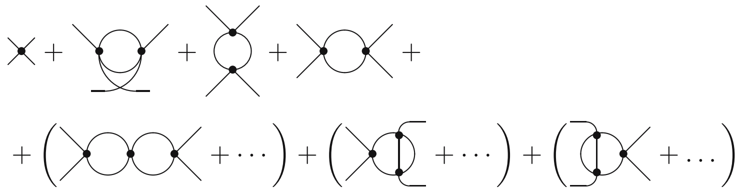
II.1. The ϕ^4 theory in one and two loops

The simplest nontrivial example is the renormalization of the four-point correlation function in the ϕ^4 theory at two loops (i.e., in the order T^3). This example is still rather simple, but it nevertheless allows illustrating the basic properties and specific features of the forest formula.

We consider the QFT defined by the Lagrangian

$$L(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{T}{4!}\phi^4. \tag{50}$$

We assume that the field and the mass term have already been renormalized. We hence do not consider renormalization of the kinetic and mass terms, i.e., bivalent vertices and divergent graphs with two external lines (see Sec. 2 in the appendix regarding them). We begin directly with the four-point function, which is written as a sum of Feynman diagrams:

$F(T) =$  (51)

The ellipses denote omitted diagrams in the u and s channels.

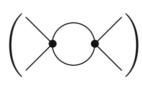
For $Z(\Gamma \,|\, T)$, we choose the functions

$Z($$\,|\, T) = -iT, \qquad Z(\text{1-loop diagram} \,|\, T) = (-iT)^2,$

$$Z(\text{2-loop diagram} \,|\, T) = (-iT)^3, \qquad \dots .$$

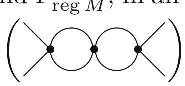
Then $\widehat{F}(\Gamma)$ are written as

$F($$) = 1,$

$F^{(1,t)}(p^2) \equiv F$  $= \dfrac{1}{2} \displaystyle\int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}.$

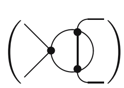
The Pauli–Villars regularization replaces $F^{(1,t)}(p^2)$ with

$$\begin{aligned} F^{(1,t)}_{\text{reg}\,M}(p^2) &= \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \left(\frac{i}{k^2 - m^2} - \frac{i}{k^2 - M^2} \right) \left(\frac{i}{(k+p)^2 - m^2} - \frac{i}{(k+p)^2 - M^2} \right) = \\ &= \frac{i}{32\pi^2} \left\{ -\log\left(\frac{M^2}{\mu^2} \right) + \int_0^1 dx \, \log \frac{m^2 - x(1-x)p^2}{\mu^2} + O\left(\frac{1}{M} \right) \right\} = \\ &= \frac{i}{32\pi^2} \left\{ -\log\left(\frac{M^2}{\mu^2} \right) + I^{(1,t)}_{\text{fin}}(p^2) \right\}. \end{aligned} \tag{52}$$

Similar formulas can also be written for $F^{(1,s)}_{\text{reg}\,M}$ and $F^{(1,u)}_{\text{reg}\,M}$; in all of them, p^2 denotes the squared momentum in the loop. The evaluation of $F^{(2,t^2)}(p^2) \equiv F$  is straightforward; it is just the square of $F^{(1,s)}$:

$$F^{(2,t^2)}_{\text{reg}\,M}(p^2) = \left(\frac{i}{32\pi^2} \right)^2 \left\{ \log^2\left(\frac{M^2}{\mu^2} \right) - 2\log\left(\frac{M^2}{\mu^2} \right) I^{(1,t)}_{\text{fin}}(p^2) + \left(I^{(1,t)}_{\text{fin}}(p^2) \right)^2 \right\}. \tag{53}$$

The other two-loop diagrams are more complicated and moreover contain overlapping divergences:

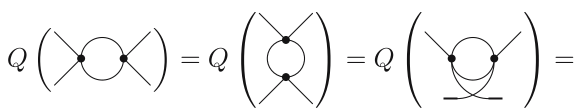
$F^{(2,t\cdot(s+u))}(p, p_4) \equiv F$  $= \displaystyle\int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2} \frac{i}{(p-q)^2 - m^2} F^{(1,t)}\big((q - p_4)^2\big).$

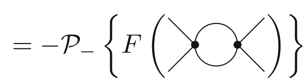
Here, q is the momentum in the larger loop, p is the total momentum passing through the larger loop, and p_4 is the momentum of the bottom-right external leg of the diagram. Substituting $F^{(1,t)}_{\text{reg}\,M}$ from (52), we obtain

$$F^{\left(2,t\cdot(s+u)\right)}_{\mathrm{reg}\,M}(p,p_4) = -2\frac{i}{32\pi^2}\log\left(\frac{M^2}{\mu^2}\right)F^{(1,t)}_{\mathrm{reg}\,M}(p^2) + \frac{i}{32\pi^2}\int\frac{d^4q}{(2\pi)^4}\left(\frac{i}{q^2-m^2}-\frac{i}{q^2-M^2}\right)\times$$
$$\times\left(\frac{i}{(q-p)^2-m^2}-\frac{i}{(q-p)^2-M^2}\right)I^{(1,s)}_{\mathrm{fin}}\left((q-p_4)^2\right) =$$
$$= \left(\frac{i}{32\pi^2}\right)^2\left\{2\log^2\left(\frac{M^2}{\mu^2}\right) - 2\log\left(\frac{M^2}{\mu^2}\right)I^{(1,s)}_{\mathrm{fin}}(p^2) -\right.$$
$$\left. -\log^2\left(\frac{M^2}{\mu^2}\right) + I^{(2,t\cdot(s+u))}_{\mathrm{fin}}(p,p_4)\right\}. \tag{54}$$

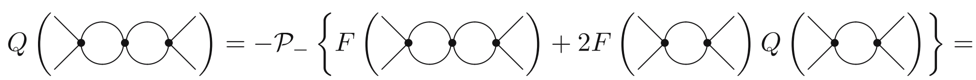
Because the expressions for each diagram naturally decompose into two parts, one finite in the limit as $M\to\infty$ and the other diverging as $\log^n(M^2/\mu^2)$ with a positive integer n, we define $\mathcal{P}_-$ such that it selects terms of the form $\log^n(M^2/\mu^2)$ with $n>0$.

We can now apply forest formula (4):

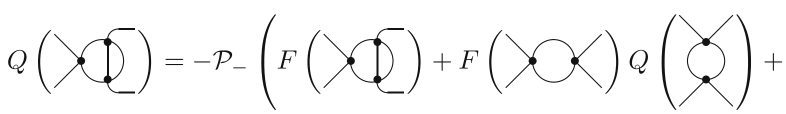


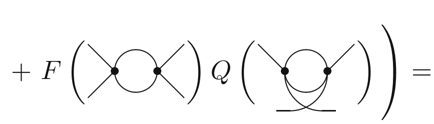


$$= \frac{i}{32\pi^2}\log\left(\frac{M^2}{\mu^2}\right), \tag{55}$$



$$= -\left[\left(\frac{i}{32\pi^2}\right)^2\left\{\log^2\left(\frac{M^2}{\mu^2}\right) - 2\log\left(\frac{M^2}{\mu^2}\right)I^{(1,s)}_{\mathrm{fin}}(p^2)\right\} +\right.$$
$$\left. + 2\times\frac{i}{32\pi^2}\left\{-\log\left(\frac{M^2}{\mu^2}\right) + I^{(1,t)}_{\mathrm{fin}}(p^2)\right\}\left\{\frac{i}{32\pi^2}\log\left(\frac{M^2}{\mu^2}\right)\right\}\right] =$$
$$= \left(\frac{i}{32\pi^2}\right)^2\log^2\left(\frac{M^2}{\mu^2}\right), \tag{56}$$





$$= -\left(\frac{i}{32\pi^2}\right)^2\mathcal{P}_-\left[\log^2\left(\frac{M^2}{\mu^2}\right) - 2\log\left(\frac{M^2}{\mu^2}\right)I^{(1,s)}_{\mathrm{fin}}(p^2) +\right.$$
$$\left. + I^{(2,t\cdot(s+u))}_{\mathrm{fin}}(p^2) - 2\left\{-\log\left(\frac{M^2}{\mu^2}\right) + I^{(1,t)}_{\mathrm{fin}}(p^2)\right\}\log\left(\frac{M^2}{\mu^2}\right)\right] =$$
$$= -3\left(\frac{i}{32\pi^2}\right)^2\log^2\left(\frac{M^2}{\mu^2}\right). \tag{57}$$

Collecting all these Q into a single formula for $\widetilde{T}$, we obtain

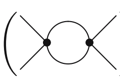
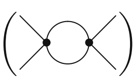
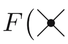
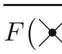
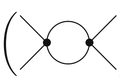
$$-i\widetilde{T} = -iT + 3\frac{i}{32\pi^2}\log\left(\frac{M^2}{\mu^2}\right)(-iT)^2 - 3\left(\frac{i}{32\pi^2}\right)^2\log^2\left(\frac{M^2}{\mu^2}\right)(-iT)^3,$$

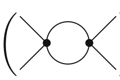
$$\widetilde{T} = T + \frac{3}{32\pi^2}\log\left(\frac{M^2}{\mu^2}\right)T^2 - \frac{3}{(32\pi^2)^2}\log^2\left(\frac{M^2}{\mu^2}\right)T^3.$$

Such $\widetilde{T}$ indeed makes $F(T)$ $\mathcal{P}$-positive, and the resulting $F(\widetilde{T})$ can again be directly obtained from second-level forest formula (5):

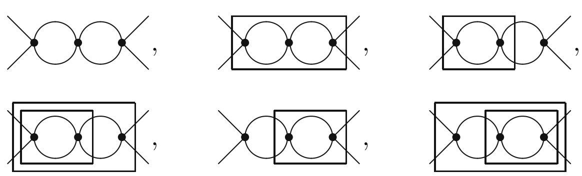
$$F_{\mathrm{R}}(T) = F(\widetilde{T}),$$

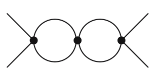
$F_{\mathrm{R}}($  $) = F($  $) = 1,$

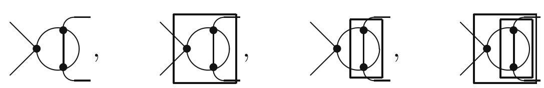
$F_{\mathrm{R}}\Big($  $\Big) = F\Big($  $\Big) + F($  $)\dfrac{1}{F(\text{ }}$  $)}(-\mathcal{P}_-)F\Big($  $\Big) =$

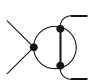
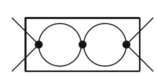
$= \mathcal{P}_+F\Big($  $\Big) = \dfrac{i}{32\pi^2}\displaystyle\int_0^1 dx\log\frac{m^2 - x(1-x)p^2}{\mu^2}.$

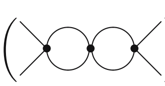
At the level of double-loop graphs, we must sum over the forests

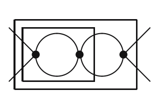


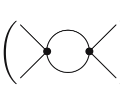
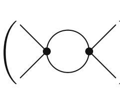
for  and over the forests



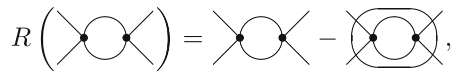
for . According to the rules described in Sec. 1.3, the forest  corresponds to

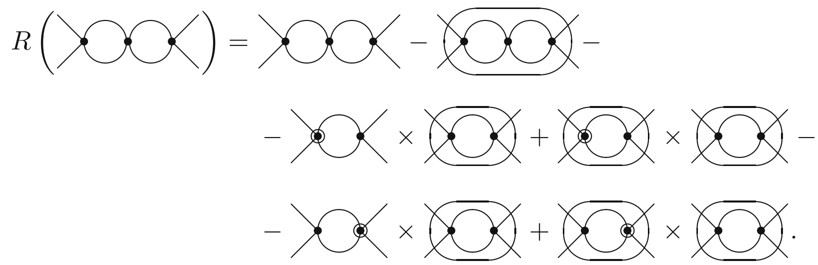
$(-\mathcal{P}_-)F\Big($  $\Big)$

(we omit $F($  $) = 1$), while  corresponds to

$(-\mathcal{P}_-)\Big\{F\Big($  $\Big)(-\mathcal{P}_-)F\Big($  $\Big)\Big\}.$

If we represent the action of $\mathcal{P}_-$ on a graph by surrounding it with an oval and let ⊙ denote the vertex that replaces a contracted graph, then we can symbolically write the expression for the $R(\Gamma) \equiv F_{\mathrm{R}}(\Gamma)$ as

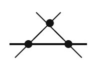

(58)


(59)

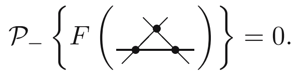
In $R(\Gamma) \equiv F_{\mathrm{R}}(\Gamma)$, it is easy to recognize the Bogoliubov R-operation [1], [2] and its graphic representation in (58) and (59).

II.2. Renormalizable and nonrenormalizable QFT models: The ϕ^4 theory in $d = 4$ and $d \geq 5$

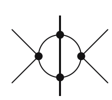
In this section, we briefly touch on the difference between renormalizable and nonrenormalizable QFT models. Because the forest formula must be applied order by order in the coupling constant, we must consider all graphs of the given order. Returning to (51) for the ϕ^4 theory, we see that there are more diagrams in the third order, for example,


(60)

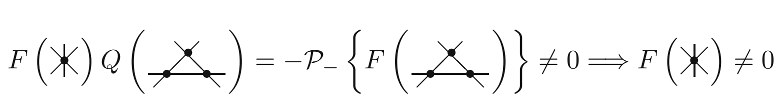
We omitted these diagrams in (51) because the expressions for them at $d = 4$ do not contain divergent terms:


(61)

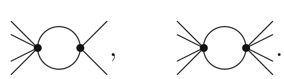
But this statement depends on the space–time dimension d, and this diagram diverges as M^{d-6} for $d \geq 6$. At $d = 5$, this graph is finite, but another diagram,


(62)

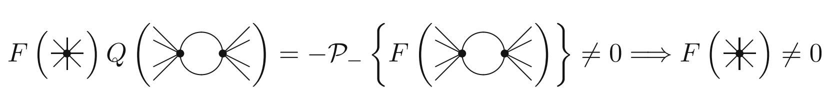
diverges. This means that to satisfy the vertex criterion and renormalize self-consistently, we must also include an elementary hexavalent vertex T_6 in $F(T)$, adding it to the elementary quadrivalent vertex T_4:



We should then consider the contributions from all diagrams that appear as a result of adding a new vertex in the given (third) order. The additional diagrams that require consideration, for example, are

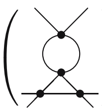


Both of them diverge because they contain a loop with two propagators, and they contribute to $Q(\Gamma)$. The first gives a correction to T_6, but the second diagram generates T_8, an elementary octavalent vertex:

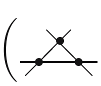
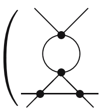
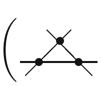
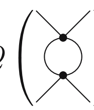


In turn, T_8 generates T_{10}, and so on. We see that for $d \geq 5$, although it is possible to renormalize $F(T)$, an infinite number of vertices should be introduced to do it self-consistently, even in the third order in T. Such theories are said to be *nonrenormalizable*.

The situation in $d = 4$ differs from that in higher dimensions. Because (61) is satisfied, diagram (60) does not contribute to the renormalization of $F(T)$ in (51) and generates no new vertices. This property is not related to the order T^3. It can be shown [1]–[3] that all divergences in graphs with more than four external legs arise from divergences of quadrivalent subgraphs. For example, although

$\mathcal{P}_- \Big\{ F \Big($  $\Big) \Big\} \neq 0$

at $d = 4$, this divergence does not require introducing the bare T_6 vertex, because it cancels in the forest formula,

$F\big($  $\big)\, Q \big($  $\big) = \mathcal{P}_- \Big\{ F \Big($  $\Big) - F \big($  $\big)\, Q \Big($  $\Big) \Big\} = 0,$

and we can consistently set $F\big($  $\big) = 0$. In the language of renormalized perturbation theory, this means that all divergences can be absorbed into a single counterterm for the quadrivalent vertex: the $Q(\Gamma)$ that makes $F(T)$ or any other correlator in our theory finite takes a nonzero value only on the graphs with four external legs. Theories with such a property are said to be *renormalizable*.

Appendix

1. Proof of the forest formula. After the large number of concrete examples that we examined in this paper, proving the forest formula becomes a simple exercise. We recall that for a given function $F(T)$ and linear projection operator $\mathcal{P}_-$, the forest formula gives a solution of Eq. (1) that allows finding the shift of variables $T \to \widetilde{T} = T + Q(T)$. In other words, the forest formula allows modifying the value of the function F as desired by shifting its argument while preserving its shape. This is important in physical applications because problems of the partition function (e.g., divergences) can be cured without violating its distinguished properties (e.g., keeping it inside the narrow class of integrable τ-functions).

Writing the representation

$$F\big(T + Q(T)\big) = \sum_{\Gamma} \widehat{F}(\Gamma) Z\big(\Gamma \,|\, T + Q(T)\big) = \sum_{\Gamma} \widehat{F}_{\mathrm{R}}(\Gamma) Z(\Gamma \,|\, T),$$

we see that because the functions $Z(\Gamma \,|\, T)$ form a basis in the space of all functions of T, Eq. (1) can be satisfied if and only if $\mathcal{P}_- F_{\mathrm{R}}(\Gamma) = 0$ for all Γ (we recall that the projection operator $\mathcal{P}_-$ does not act on the $Z(\Gamma \,|\, T)$). To find the coefficients $\widehat{F}_{\mathrm{R}}(\Gamma)$, we substitute $Q(T) = \sum_\gamma \widehat{Q}(\gamma) Z(\gamma \,|\, T)$ and compare terms in the two sides of the equality

$$\sum_{\Gamma} \widehat{F}(\Gamma) Z\bigg(\Gamma \,\bigg|\, T + \sum_{\gamma} \widehat{Q}(\gamma) Z(\gamma \,|\, T)\bigg) = \sum_{\Gamma} \widehat{F}_{\mathrm{R}}(\Gamma) Z(\Gamma \,|\, T).$$

Because

$$Z\bigg(\Gamma \,\bigg|\, T + \sum_{\gamma} \widehat{Q}(\gamma) Z(\gamma \,|\, T)\bigg) = Z(\Gamma \,|\, T) + \sum_{\Upsilon, \gamma:\ \Upsilon/\gamma = \Gamma} \widehat{Q}(\gamma) Z(\Upsilon \,|\, T) + \\ + \sum_{\Upsilon, \gamma_1, \gamma_2:\ \Upsilon/(\gamma_1\gamma_2) = \Gamma} \widehat{Q}(\gamma_1)\widehat{Q}(\gamma_2) Z(\Upsilon \,|\, T) + \ldots,$$

we obtain

$$\widehat{F}_{\mathrm{R}}(\Gamma) = \widehat{F}(\Gamma) + \sum_{\{\gamma_1 \cup \cdots \cup \gamma_k\}} \widehat{F}(\Gamma/(\gamma_1 \cdots \gamma_k))\widehat{Q}(\gamma_1) \cdots \widehat{Q}(\gamma_k),$$

$$\mathcal{P}_- \widehat{F}_{\mathrm{R}}(\Gamma) = 0, \tag{A.1}$$

$$\mathcal{P}_- \Big\{ \widehat{F}(\Gamma) + \widehat{F}(\Gamma/\Gamma)\widehat{Q}(\Gamma) + \sum_{\{\gamma_1 \cup \cdots \cup \gamma_k\}}{}' \widehat{F}(\Gamma/(\gamma_1 \cdots \gamma_k))\widehat{Q}(\gamma_1) \cdots \widehat{Q}(\gamma_k) \Big\} = 0,$$

where the prime on the summation symbol means that we extracted the term with $\gamma = \Gamma$ from the sum. Assuming that $\mathcal{P}_+ \widehat{F}(\Gamma/\Gamma)\widehat{Q}(\Gamma) = 0$, we now obtain forest formula (4),

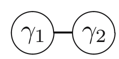
$$\widehat{F}(\Gamma/\Gamma)\widehat{Q}(\Gamma) = -\mathcal{P}_- \Big\{ \widehat{F}(\Gamma) + \sum_{\{\gamma_1 \cup \cdots \cup \gamma_k\}} \widehat{F}(\Gamma/(\gamma_1 \cdots \gamma_k))\widehat{Q}(\gamma_1) \cdots \widehat{Q}(\gamma_k) \Big\},$$

which was to be proved.

Second-level forest formula (5) can be obtained by solving the recursion for $\widehat{Q}(\Gamma)$ and substituting the result in (A.1).

2. Kinetic terms, 1PR diagrams, and other specific features. Here we briefly comment on specific features that were not considered in the somewhat oversimplified treatment in Sec. II.1.

The first point to explain is the situation with the 1PR (one-particle-reducible) graphs. They were excluded from the treatment in Sec. II.1 because they do not contribute to the counterterms Q. This can be easily proved by induction. Indeed, we consider a 1PR graph

$\Gamma =$ 

(external legs are not shown). Its Feynman amplitude is given by the product of the propagator and two amplitudes for its 1PI subgraphs γ_1 and γ_2. Assuming that Q for all the 1PR graphs with a smaller number of vertices is zero, we conclude that the set of box subgraphs of Γ that can potentially contribute to $Q(\Gamma)$ is just the union of the corresponding sets for its 1PI subgraphs and the boxes that entirely contain γ_1 or γ_2. We thus obtain

$$Q(\Gamma) = \mathcal{P}_- \big\{ \big(F(\gamma_1) - Q(\gamma_1)\big)\big(F(\gamma_2) - Q(\gamma_2)\big) \big\} \times \{ \text{—} \} = 0$$

because

$$\mathcal{P}_- \big\{ F(\gamma_1) - Q(\gamma_1) \big\} = \mathcal{P}_- \big\{ F(\gamma_2) - Q(\gamma_2) \big\} = 0.$$

This derivation is clearly based on the fact that all divergences in QFT come from loops, and the 1PR graph is just the product of its 1PI components.

The second point to clarify concerns the renormalization of the mass and kinetic terms. To study this subject, we must introduce two more coupling constants in our treatment in Sec. II.1 and correspondingly two more elementary vertices, —•— and —×—, respectively corresponding to the mass term and to the kinetic term. The Lagrangian and elementary Feynman vertices are now

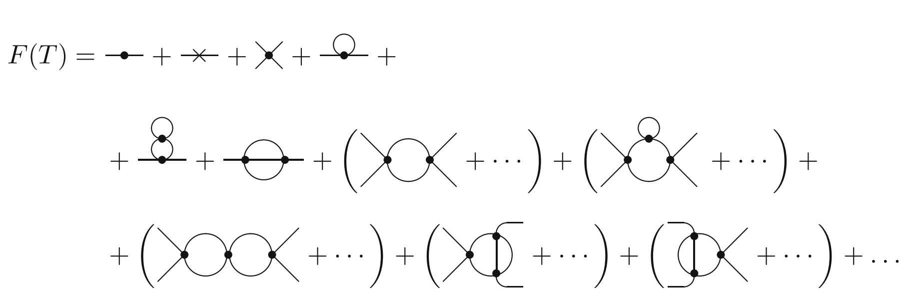
$$L(\phi) = \frac{\kappa}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{T}{4!}\phi^4,$$

$$F(\text{—•—}) = 1, \qquad F(\text{—×—}) = p^2, \qquad F(\times) = 1,$$

$$Z(\text{—•—} \,|\, T) = m^2, \qquad Z(\text{—×—} \,|\, T) = \kappa, \qquad Z(\times \,|\, T) = -iT.$$

It is usually assumed that the physical value $\tilde{\kappa} = 1$, but its bare value κ can be different.

We can now write the sum of all correlators that can potentially contribute to Q, i.e., potentially diverge:



Because we have two different elementary vertices corresponding to two different coupling constants, a new external structure should be introduced to separate contributions to Q into two parts: the Q for graphs with two external legs acquires an additional label, either 0 or 1. We have two projection operators $\mathcal{P}_-^{(0)}$ and $\mathcal{P}_-^{(1)}$ for the self-energy graphs: $\mathcal{P}_-^{(0)}$ selects the singular terms proportional to p^0, and $\mathcal{P}_-^{(1)}$ selects the terms proportional to p^2. Accordingly,

$$Q^{(i)}(\Gamma) = -\mathcal{P}_-^{(i)}\big\{F(\Gamma) + \dots\big\}.$$

We can now renormalize "as usual":

$$\widetilde{m}^2 = m^2 + \sum_{\substack{\text{self-energy} \\ \text{graphs}}} Q^{(0)}(\Gamma),$$

$$\tilde{\kappa} = \kappa + \sum_{\substack{\text{self-energy} \\ \text{graphs}}} Q^{(1)}(\Gamma),$$

$$-i\widetilde{T} = -iT + \sum_{\substack{\text{graphs with 4} \\ \text{external legs}}} Q(\Gamma).$$

A more elaborated treatment of renormalization with mass terms can be found in [6], [8] for the ϕ^3 theory, in [5] for the ϕ^4 theory, and in [8], [9] for quantum electrodynamics.

Acknowledgments. This work is supported in part by the Dynasty Foundation (M. N. S.), the Russian Foundation for Basic Research (Grant Nos. 07-02-00878 and 07-02-00645), a joint grant (Grant No. 06-01-92059-CE), the NWO (Project No. 047.011.2004.026), INTAS (Grant No. 05-1000008-7865), the Program for Supporting Leading Scientific Schools (Grant No. NSh-8004.2006.2), and also by a project (Project No. ANR-05-BLAN-0029-01, A. Yu. M.).

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